TP A.30
The effect of cue tip offset, cue weight, and cue speed on cue ball speed and spin

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"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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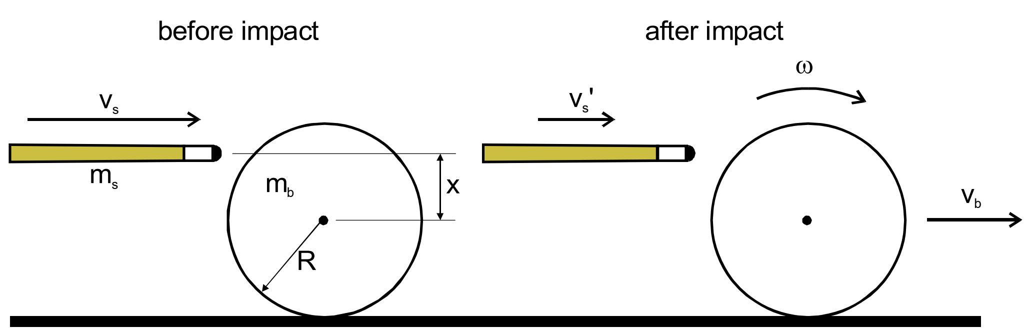


Linear momentum must be conserved during the impact, so:

$$m_s v_s = m_s v_s' + m_b v_b \tag{1}$$

If an elastic collision is assumed, energy is also conserved:

$$\frac{1}{2} m_s v_s^2 = \frac{1}{2} m_s v_s'^2 + \frac{1}{2} m_b v_b^2 + \frac{1}{2}\left(\frac{2}{5} m_b R^2\right)\omega^2 \tag{2}$$

The linear impulse between the tip and ball at impact is equal to the change in momentum of both the stick and ball:

$$\hat{F} = m_s\left(v_s - v_s'\right) = m_b v_b \tag{3}$$

and the angular impulse is equal to the change in angular momentum of the ball:

$$x\hat{F} = \left(\frac{2}{5} m_b R^2\right)\omega \tag{4}$$

From Equation 1, the final stick speed can be expressed as:

$$v_s' = v_s - \frac{m_b}{m_s} v_b \tag{5}$$

and from Equations 3 and 4, the spin rate of the ball can be expressed as:

$$\omega = \frac{5}{2} \frac{v_b}{R^2} x \tag{6}$$

Substituting Equations 5 and 6 into Equation 2, and solving for the final ball speed gives:

$$v_b = \frac{2v_s}{\left[1 + \frac{m_b}{m_s} + \frac{5}{2}\left(\frac{x}{R}\right)^2\right]} \tag{7}$$

Obviously, to achieve maximum ball speed (v_b) for a given stick speed (v_s), the offset (x) should be 0 (i.e., center-ball hit). For a typical mass ratio (1/3, per below), the maximum ball speed is related to the stick speed (neglecting energy losses) according to:

$$\frac{v_b}{v_s} = \frac{2}{\left[1 + \frac{1}{3}\right]} = \frac{3}{2} = 1.5 \tag{8}$$

So for a center-ball hit, with typical equipment (and neglecting energy losses), the ball moves 50% faster than the stick. With an extremely heavy stick ($m_s >> m_b$), the theoretical limit for the maximum speed of the ball is twice the speed of the stick.

Substituting Equation 7 into Equation 6 gives the ball spin rate:

$$\omega = \frac{5v_s x}{R^2\left[1 + \frac{m_b}{m_s} + \frac{5}{2}\left(\frac{x}{R}\right)^2\right]} = \frac{N}{D} \tag{9}$$

For maximum ball spin, for a given cue-stick speed, the derivative of ω with respect to x must be zero, so:

$$\frac{d\omega}{dx} = 0 \quad \Rightarrow \quad DN' - ND' = R^2\left[1 + \frac{m_b}{m_s} + \frac{5}{2}\left(\frac{x}{R}\right)^2\right]5v_s - 5v_s x R^2\left[\frac{5}{R^2}x\right] = 0 \tag{10}$$

Solving this equation for x gives the tip offset that results in the maximum ball spin-rate:

$$\frac{x}{R} = \sqrt{\frac{2}{5}\left(1 + \frac{m_b}{m_s}\right)} \tag{11}$$

Typical mass values are 6 oz for the ball and 18 oz for the cue stick. This gives a mass ratio of 1/3, so the predicted offset for maximum spin is:

$$\frac{x}{R} = \sqrt{\frac{8}{15}} \approx 0.73 \tag{12}$$

When considering the offset for maximum spin, we need to ensure the ball separates from the cue stick after impact; otherwise, the tip would continue to push and rub on the cue ball after the initial impact (if we neglect the effects of squirt and cue stick deflection and vibration for the moment). The condition for separation between the tip and ball is:

$$v_b > v_s' \tag{13}$$

Equations 5 and 7 can be used to show that this constraint is equivalent to:

$$\frac{x}{R} < \sqrt{\frac{2}{5}\left(1 + \frac{m_b}{m_s}\right)} \tag{14}$$

Comparing this to Equation 11 shows that the offset that produces the maximum spin rate is also on the verge of preventing cue tip and ball separation. So it would appear that 0.73R is still the optimal value for achieving maximum ball spin.

The analysis above has assumed the collision is perfectly elastic (i.e., no energy is lost). Now, we'll look at the effects of an inelastic collision between the cue tip and cue ball. In this case, the coefficient of restitution (e) is less than one, and energy is lost in the collision. If η represents the efficiency of the collision, Equation 2 can be written as:

$$\eta \frac{1}{2} m_s v_s^2 = \frac{1}{2} m_s v_s'^2 + \frac{1}{2} m_b v_b^2 + \frac{1}{2}\left(\frac{2}{5} m_b R^2\right)\omega^2 \tag{15}$$

This equation can be written (by multiplying by $2/m_s$) as:

$$\eta v_s^2 = v_s'^2 + m_r v_b^2 + \frac{2}{5} m_r R^2 \omega^2 \tag{16}$$

where m_r is the ball-to-cue-stick mass ratio:

$$m_r = \frac{m_b}{m_s} \tag{17}$$

Substituting Equations 5 and 6 into Equation 16 and simplifying results in:

$$\left[1 + m_r + \frac{5}{2}\left(\frac{x}{R}\right)^2\right] v_b^2 + \left[-2v_s\right] v_b + \left[\frac{1}{m_r}(1-\eta) v_s^2\right] = 0 \tag{18}$$

Equation 18 is a quadratic, and can be solved for the final ball speed:

$$v_b = v_s \left\{ \frac{1 + \sqrt{\eta - \frac{1-\eta}{m_r}\left[1 + \frac{5}{2}\left(\frac{x}{R}\right)^2\right]}}{\left[1 + m_r + \frac{5}{2}\left(\frac{x}{R}\right)^2\right]} \right\} \tag{19}$$

For an elastic collision (e = 1), no energy is lost (η=1=100%), and Equation 19 reduces to Equation 7.

To see how the collision efficiency (η) is related to the coefficient of restitution (e), let's look at the simple case of a center-ball hit (x=0, ω=0). The coefficient of restitution (COR = e) is defined as the ratio of the speed of separation and the speed of approach between the cue stick and cue ball:

$$e = \frac{speed\ of\ separation}{speed\ of\ approach} = \frac{v_b - v_s'}{v_s} \tag{20}$$

From Equations 1 and 20, it can be shown that final cue stick speed is

$$v_s' = \frac{(1 - m_r e)}{(1 + m_r)} v_s \tag{21}$$

and the final cue ball speed is:

$$v_b = \frac{(1 + e)}{(1 + m_r)} v_s \tag{22}$$

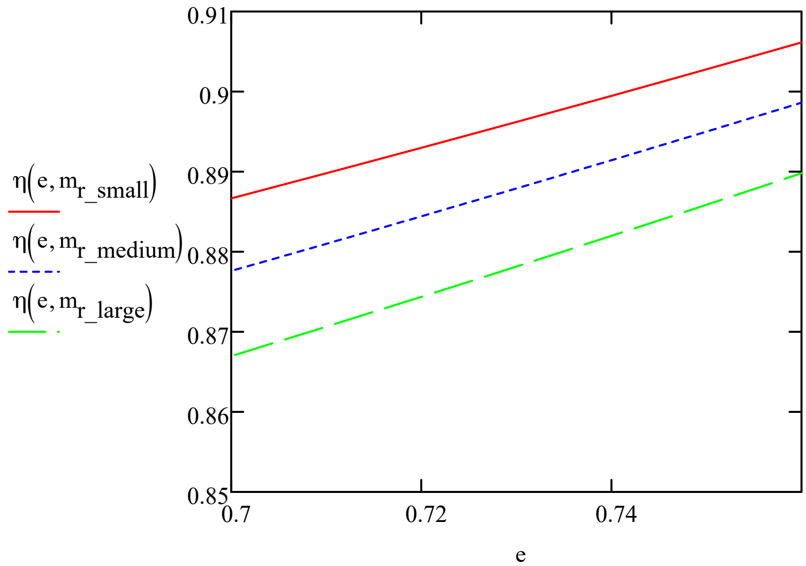
Substituting Equations 21 and 22 into Equation 16 (with ω=0), gives:

$$\eta = \frac{(1 - m_r e)^2 + m_r (1 + e)^2}{(1 + m_r)^2} \tag{23}$$

Here's a plot of how the efficiency η varies with e and m_r, for typical ranges of values:

$$\eta(e, m_r) := \frac{(1 - m_r \cdot e)^2 + m_r \cdot (1 + e)^2}{(1 + m_r)^2}$$

$e := 0.68, 0.69 .. 0.76 \quad m_{r\_small} := \frac{6}{21} \quad m_{r\_medium} := \frac{6}{19} \quad m_{r\_large} := \frac{6}{17}$



The efficiency and COR vary with cue tip type and the mass of the cue stick. The COR (e) also might vary with tip offset and shot speed, but I don't have data to quantify this. From HSV B.42, typical leather-tip playing-cue CORs are in the 0.73 range. For a 19 oz cue, that corresponds to an efficiency of:

$$\eta\left(0.73, \frac{6}{19}\right) = 0.888$$

For the remainder of the analysis, I'll use an efficiency of 0.87 (87%), corresponding to an energy loss of 13%, which is fairly typical and is what Coriolis used in his book.

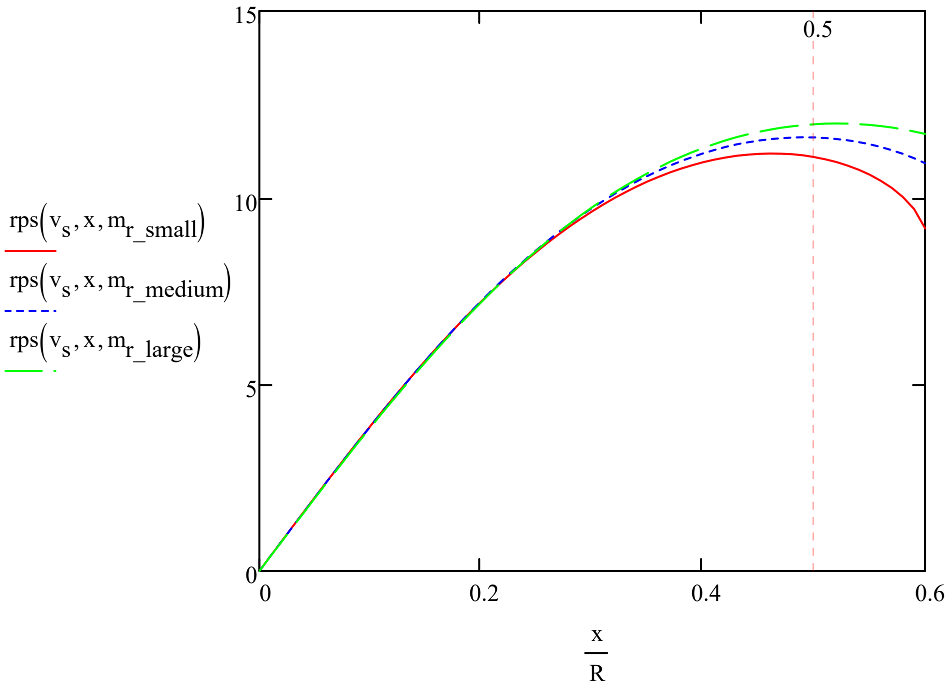
$$\eta := 0.87$$

We will now look at how ball speed (from Equation 19) and spin (from Equation 6, expressed in units of revolutions per second), vary with tip offset and cue mass for a constant collision efficiency:

$R := \frac{2.25 \cdot in}{2}$ ball radius $\qquad v_s := 5 \cdot mph$ typical stick speed

$$v_b(v_s, x, m_r) := v_s \cdot \frac{1 + \sqrt{\eta - \frac{1 - \eta}{m_r} \cdot \left[1 + \frac{5}{2} \cdot \left(\frac{x}{R}\right)^2\right]}}{\left[1 + m_r + \frac{5}{2} \cdot \left(\frac{x}{R}\right)^2\right]} \qquad rps(v_s, x, m_r) := \frac{1}{2 \cdot \pi} \cdot \frac{5}{2} \cdot v_b(v_s, x, m_r) \cdot \frac{x}{R^2}$$

$$x := 0\cdot R, 0.01\cdot R .. 0.6\cdot R$$



Therefore, with a typical collision efficiency, the offset that produces the most spin is close to the miscue limit (0.5). Note that, for the assumptions above, a lighter cue (with a larger mass ratio m_r), will deliver slightly more spin to the cue ball than a heavier cue for offsets close to the miscue limit (x/R = 0.5), but this probably isn't enough of a reason to try to switch to a lighter cue stick. The only other way to increase the amount of spin, other than by increasing the tip offset (up to the miscue limit), is to increase the stick speed.

It is interesting to compare the relative effects of increasing stick speed vs. increasing tip offset. Let's look at increasing stick speed by 10% vs. increasing the offset by 10% (for 50% and 100% maximum offset, and at medium speed):

$$m_r := \frac{6}{19} \qquad \text{typical ball-to-stick mass ratio}$$

$$x := 0.25\cdot R \qquad \frac{rps(1.1v_s, x, m_r)}{rps(v_s, x, m_r)} = 1.1 \qquad \frac{rps(v_s, 1.1x, m_r)}{rps(v_s, x, m_r)} = 1.069$$

$$x := 0.5\cdot R \qquad \frac{rps(1.1v_s, x, m_r)}{rps(v_s, x, m_r)} = 1.1 \qquad \frac{rps(v_s, 1.1x, m_r)}{rps(v_s, x, m_r)} = 0.984$$

So increasing the stick speed has the same effect on the spin rate, regardless of the amount of offset. The spin rate increases by the same amount as the speed. However, increasing the tip offset has a smaller effect, and this effect decreases at larger offsets. Therefore, it is safer to keep the cue tip away from the miscue offset limit a little, and just add more stick speed to get more spin. This is more efficient than trying to add more offset close to the limit, provided you can create more stick speed without sacrificing accuracy too much.

Now let's look at how the cue stick and cue ball speeds after impact vary with tip offset for different ball-to-cue mass ratios.

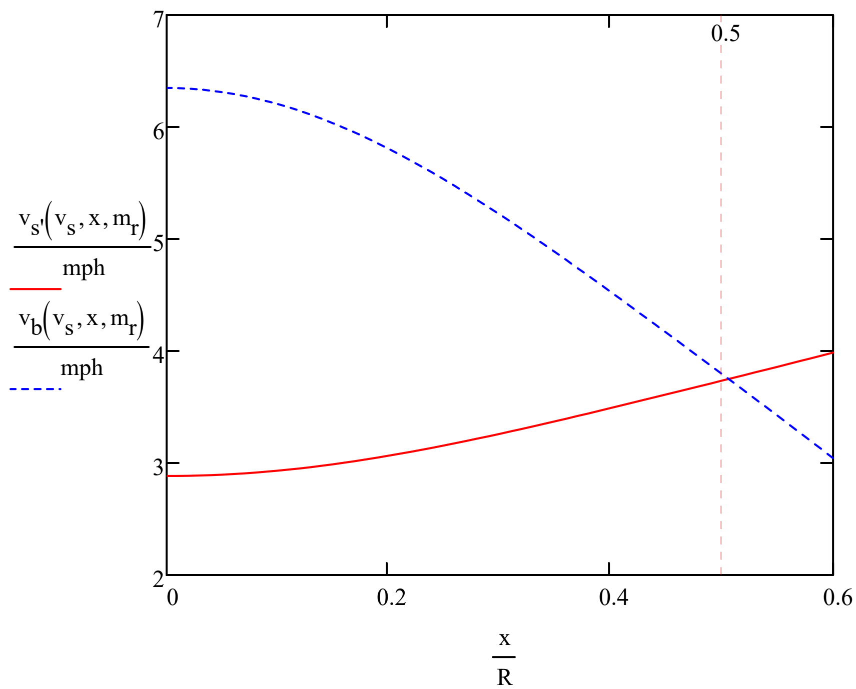
From Equation 5, the cue stick speed after impact ($v_{s'}$) is related to the cue ball speed (v_b) according to:

$$v_{s'}(v_s, x, m_r) := v_s - m_r \cdot v_b(v_s, x, m_r)$$

First, let's look at a medium-weight cue (18 oz):

$$m_r := \frac{6 \cdot oz}{18 \cdot oz}$$

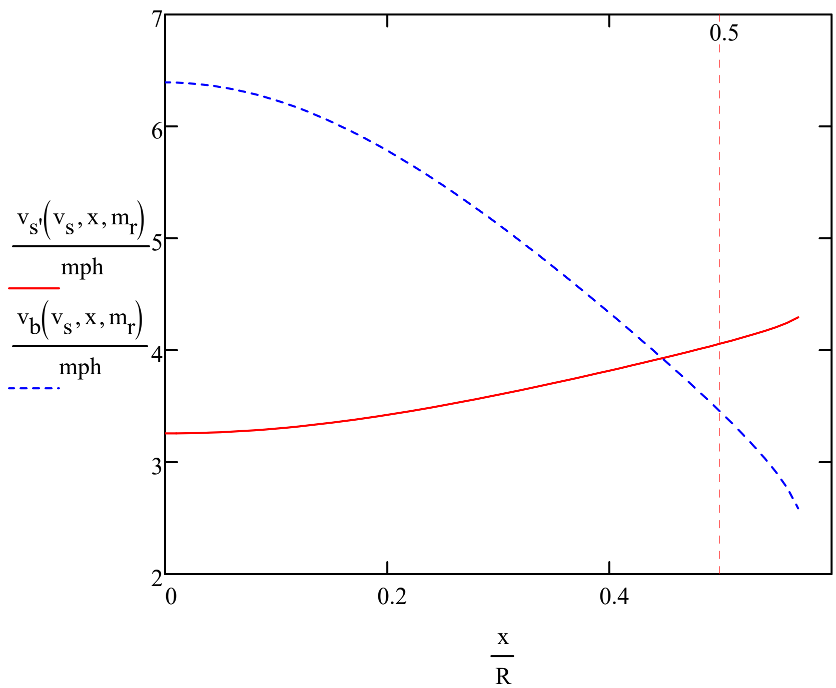
$$x := 0 \cdot R, 0.01 \cdot R .. 0.6 \cdot R$$



Therefore, close to the miscue limit (x/R = 0.5) the cue ball is at risk of not separating from the cue tip cleanly. Although, in reality, the cue tip deflects away from the cue ball after the hit, likely giving the cue ball time and distance to clear.

Now let's look at a heavier cue (22 oz):

$$m_r := \frac{6 \cdot oz}{22 \cdot oz}$$



Therefore, with a heavier cue, the cue ball will definitely not separate cleanly from the cue tip for offsets greater than about x/R=0.45, and double hits, pushes, and/or miscues would be likely.

Now let's look at how cue ball speed varies with stick weight and collision efficiency.

Remember, from Equation 23, collision efficiency depends on both the coefficient of restitution (COR or e) and the ball-to-stick mass ratio (m_r):

$$\eta(e, m_r) := \frac{(1 - m_r \cdot e)^2 + m_r \cdot (1 + e)^2}{(1 + m_r)^2}$$

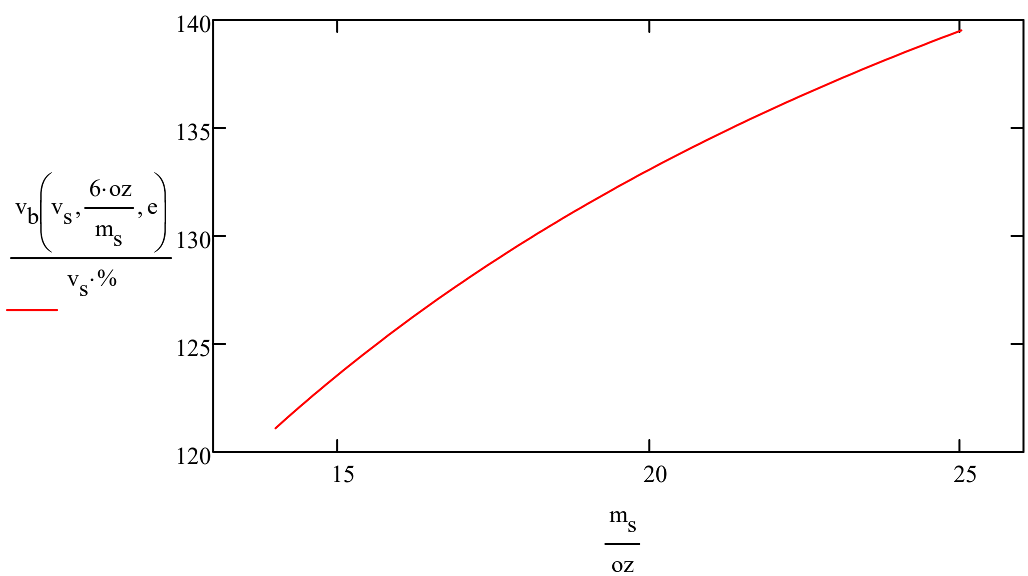
From Equation 19, for a center-ball hit (x=0), the cue ball speed depends on the stick speed (v_s), the ball-to-stick mass ratio (m_r), and the COR (e) according to:

$$v_b(v_s, m_r, e) := v_s \cdot \frac{1 + \sqrt{\eta(e, m_r) - \frac{1 - \eta(e, m_r)}{m_r}}}{(1 + m_r)}$$

The following plot shows how cue ball speed (as a percentage of stick speed) varies over a wide range of possible cue weights. A typical value is used for the COR (0.73), based on HSV B.42 with a typical playing cue with a medium-hardness leather tip. This corresponds to an efficiency of about 89% for a typical-weight cue (19 oz):

$$e := 0.73 \qquad m_r := \frac{6 \cdot oz}{19 \cdot oz} \qquad \eta(e, m_r) = 88.79 \cdot \%$$

$$v_s := 15 \cdot mph \qquad m_s := 14 \cdot oz, 14.1 \cdot oz .. 25 \cdot oz$$

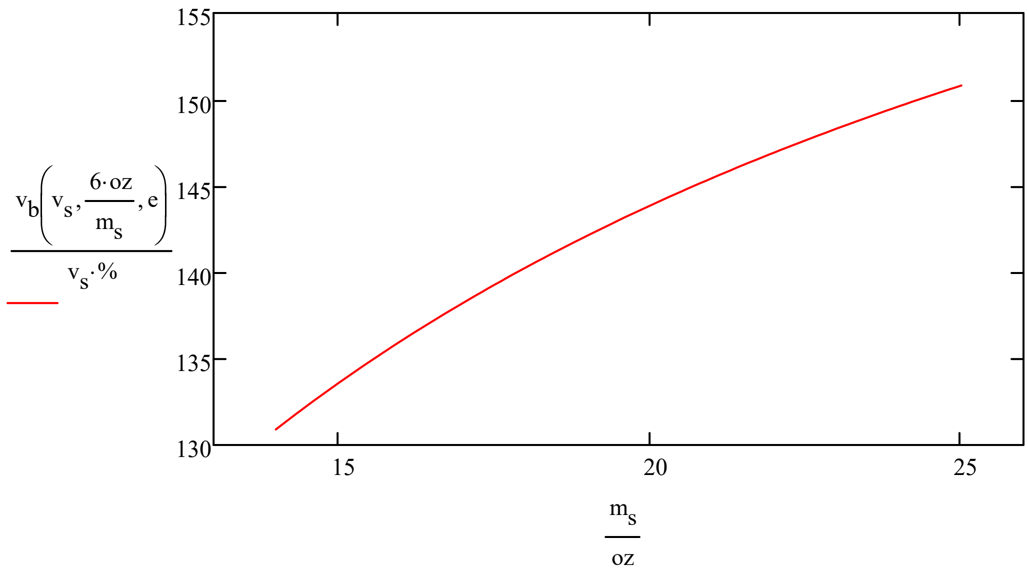


Notice that for a given cue speed, more cue weight gives more cue ball speed, as one would expect. Although, the benefit of the added weight diminishes at higher weights. Also, in reality, it is more difficult to stroke a heavier cue at the same speed as a lighter cue.

With a higher-efficiency collision, the results change a little. From HSV B.42, typical phenolic-tip break cues have a COR (e) as high as 0.87. For a 19oz cue, that corresponds to a collision efficiency of about 94%:

$$e := 0.87 \qquad m_r := \frac{6}{19} \qquad \eta(e, m_r) = 94.166\cdot\%$$

$$v_s := 15\cdot mph \qquad m_s := 14\cdot oz, 14.1\cdot oz \,..\, 25\cdot oz$$



With a perfectly elastic collision (100% efficient), the theoretical limit for the ball-speed-to-stick-speed percentage, for a ridiculously-heavy cue is 200%, as predicted by Equation 7. This is not realistic to achieve, but one thing is clear: a higher-efficiency tip results in significantly more cue ball energy. Here are typical values comparing a leather-tip playing cue to a phenolic-tip break cue:

$$e_{leather} := 0.73 \qquad e_{phenolic} := 0.87 \qquad v_s := 15mph \qquad m_r := \frac{6\cdot oz}{19\cdot oz}$$

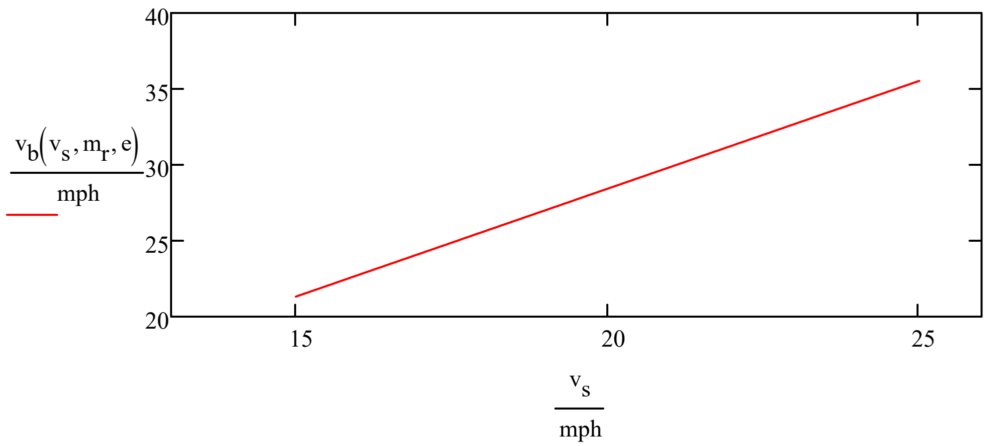
Breaking power is related to the energy delivered to the cue ball, which is related to the square of the speed, so here is how the tips compare in terms of breaking power:

$$\frac{v_b(v_s, m_r, e_{phenolic})^2}{v_b(v_s, m_r, e_{leather})^2} = 1.168$$

So a high-efficiency phenolic tip provides about a 17% increase in breaking power over a typical medium-hardness leather tip, which is significant.

For a given cue weight, cue ball speed increases linearly with cue speed, as shown by the following plot (and as predicted by Equation 7):

$m_s := 19 \cdot oz \qquad v_s := 15 \cdot mph, 15.1 \cdot mph .. 25mph \qquad m_r := \frac{6 \cdot oz}{m_s} \qquad e := 0.87$



If you could increase in your cue speed by 10%, for a given weight cue, the cue ball speed would also increase by 10%, and the effective increase in breaking power would be 21%:

$$\frac{(1.1 \cdot v_b)^2}{v_b^{\,2}} = 1.21$$

If you were able to generate the same cue speed with a range of cue weights, you can also increase breaking performance by using a heavier cue. Increasing cue weight from 17 to 22 oz (with a fixed cue speed) would increase the cue ball speed by 6.3%, which would correspond to an effective increase in breaking power of 13%:

$v_s := 15 \cdot mph \qquad m_{r\_light} := \frac{6 \cdot oz}{17oz} \qquad m_{r\_heavy} := \frac{6 \cdot oz}{22oz}$

$$\frac{v_b(v_s, m_{r\_heavy}, e)}{v_b(v_s, m_{r\_light}, e)} = 1.063 \qquad \frac{v_b(v_s, m_{r\_heavy}, e)^2}{v_b(v_s, m_{r\_light}, e)^2} = 1.13$$

So with a dramatic increase in cue weight (17 oz to 22 oz), the benefit is not as large as one might expect, even if the heavier cue could be stroked at the same speed as the lighter cue (which is usually not the case).

So what break cue weight should you use to get the most power with the best tip? Obviously, and as shown above, a heavier cue will deliver more speed to the cue ball; however, it is more difficult to generate stick speed with a heavier cue. The optimal cue weight, allowing for the best combination of weight and speed will vary with each individual based on muscle physiology. For more info, see the cue weight FAQ page here:

https://billiards.colostate.edu/faq/cue/weight/