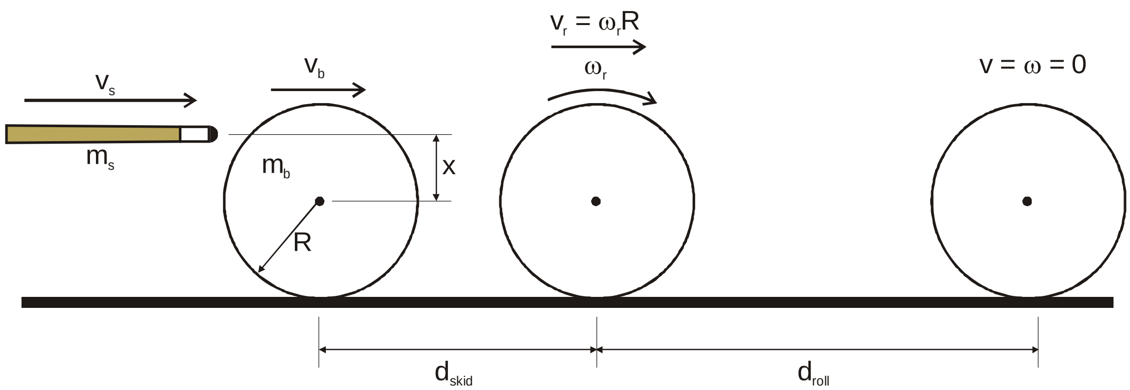

technical proof


technical proof

TP B.12
Optimal Tip Height for Speed/Distance Control

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
by David G. Alciatore, PhD, PE ("Dr. Dave")

originally posted: 8/6/2011 last revision: 8/6/2011



Relevant physical parameters (from "physics" FAQ page):

$\mu_s := 0.2$	typical ball-cloth coefficient of sliding friction
$\mu_r := 0.01$	typical ball-cloth coefficient of rolling resistance
$\eta := 0.87$	typical cue tip efficiency
$R := 1.125 \cdot in$	ball radius
$m_r := \frac{6}{19}$	typical ball-mass-to-cue-mass ratio (m_b/m_s) [for a 19 oz cue]
$x_{max} := \frac{R}{2}$	safe miscue limit
$v_s := 3 \cdot mph$	typical slow cue speed

From TP A.30, accounting for tip inefficiency, the CB speed and spin after impact are:

$$v(v_s, x, \eta, m_r) := v_s \cdot \frac{1 + \sqrt{\eta - \frac{1-\eta}{m_r} \cdot \left[1 + \frac{5}{2} \cdot \left(\frac{x}{R}\right)^2\right]}}{\left[1 + m_r + \frac{5}{2} \cdot \left(\frac{x}{R}\right)^2\right]} \qquad \omega(v_s, x, \eta, m_r) := \frac{5}{2} \cdot v(v_s, x, \eta, m_r) \cdot \frac{x}{R^2}$$

From TP B.5, the distance required for a skidding (sliding) CB to develop natural roll, and the speed when roll first develops, are:

$$d_{skid}(v, \omega, \mu_s) := \frac{2}{49 \cdot \mu_s \cdot g} \cdot \left[6 \cdot v^2 - 5 v \cdot R \cdot \omega - (R \cdot \omega)^2\right]$$

$$v_{skid}(v, \omega) := \frac{5}{7} \cdot v + \frac{2}{7} \cdot R \cdot \omega$$

From TP 4.1, the distance required for a rolling ball to stop (assuming no cushion contact) is:

$$d_{roll}(v, \mu_r) := \frac{v^2}{2 \cdot \mu_r \cdot g}$$

Therefore, the total ball travel distance (assuming no cushion contact) for a given cue speed is:

$$\begin{aligned} d(v_s, x, \eta, m_r, \mu_s, \mu_r) := {} & d_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r), \mu_s) \ \ldots \\ & + d_{roll}(v_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r)), \mu_r) \end{aligned}$$

Comparing typical skid and roll distances for a center-ball hit:

$$x := 0 \qquad v(v_s, x, \eta, m_r) = 3.824\ \text{mph}$$

$$\omega(v_s, x, \eta, m_r) = 0 \cdot \text{rpm} \qquad v_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r)) = 2.731 \cdot \text{mph}$$

$$d_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r), \mu_s) = 1.197 \cdot \text{ft}$$

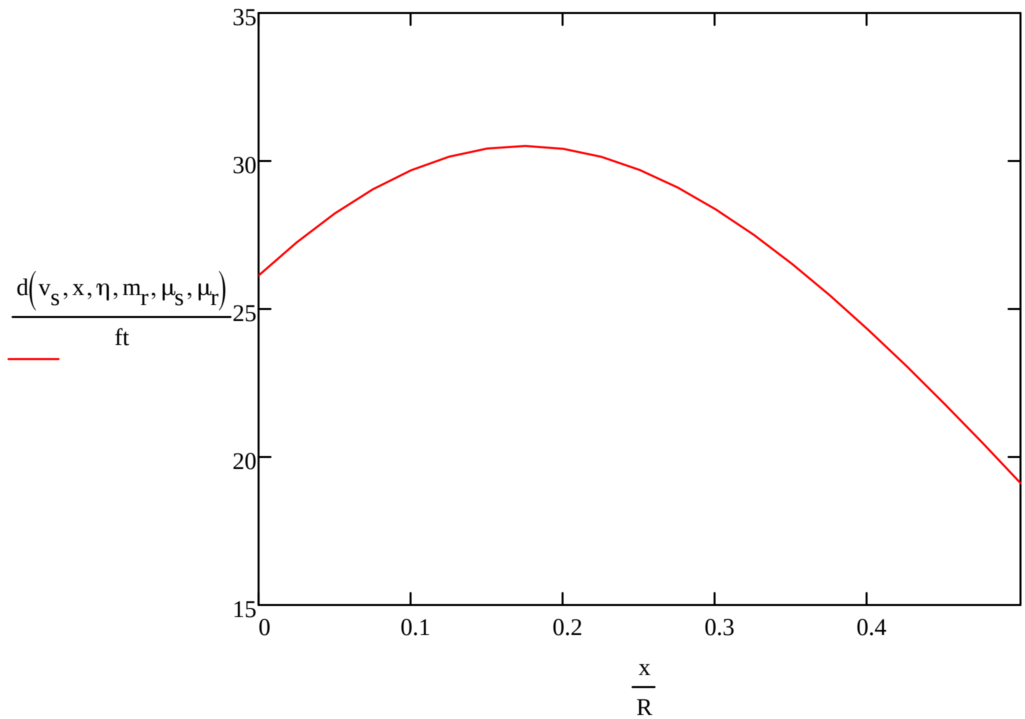
$$d_{roll}(v_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r)), \mu_r) = 24.935 \cdot \text{ft}$$

$$\frac{d_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r), \mu_s)}{d_{roll}(v_{skid}(v(v_s, x, \eta, m_r), \omega(v_s, x, \eta, m_r)), \mu_r)} = 4.8 \cdot \%$$

So the the distance traveled, with typical values, depends mostly on the "speed" (rolling resistance) of the cloth and not the "slickness" (sliding friction).

Now let's look at how ball travel distance varies with tip contact-point height, for a given cue speed:

$$x := 0, 0.05 \cdot x_{max} .. x_{max}$$



$x := 0 \quad f(x) := d(v_s, x, \eta, m_r, \mu_s, \mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.174 \quad \frac{x}{x_{max}} = 34.864 \cdot \%$

With typical conditions, the optimal tip height appears to be about 17.5% of the ball radius or about 35% of the maximum offset. At this tip height, the ball travel distance is least sensitive to slight errors in tip height. So, for a given cue speed, the ball travel distance will be more consistent from one shot to the next (with slight variance in tip height relative to the optimum). As would be expected, the optimal tip height is the same regardless of the cue speed (although the travel distance is obviously different for different cue speeds):

$f(x) := d(2 \cdot mph, x, \eta, m_r, \mu_s, \mu_r) \qquad x := \text{Maximize}(f, x)$

$\frac{x}{R} = 0.174 \qquad \frac{x}{x_{max}} = 34.864 \cdot \% \qquad f(x) = 13.559\,ft$

$f(x) := d(3 \cdot mph, x, \eta, m_r, \mu_s, \mu_r) \qquad x := \text{Maximize}(f, x)$

$\frac{x}{R} = 0.174 \qquad \frac{x}{x_{max}} = 34.864 \cdot \% \qquad f(x) = 30.508\,ft$

Ron Shepard, in his "Amateur Physics for the Amateur Pool Player" book, did a similar analysis (see Problem 3.11), but instead of calculating the optimal tip height for ball travel distance, he did so for the post-drag rolling speed. He neglected the drag (skid) distance, and he also assumed a perfectly elastic cue tip. These assumptions can be modeled in this analysis with a tip efficiency of 100% and really sticky cloth (high sliding friction):

$$\eta := 1 \qquad \mu_s := 1000$$

In Shepard's analysis, he maximized the ratio between the post-skid rolling velocity and the cue speed:

$$VR(x) := \frac{v_{skid}\left(v\left(v_s, x, \eta, m_r\right), \omega\left(v_s, x, \eta, m_r\right)\right)}{v_s}$$

$$x := \text{Maximize}(VR, x) \qquad \frac{x}{R} = 0.235 \qquad \frac{x}{x_{max}} = 47.088 \cdot \%$$

This agrees with the result from Shepard's equations:

$$VR_{Shepard}(x) := \frac{10}{7}\left[\frac{1 + \frac{x}{R}}{1 + m_r + \frac{5}{2}\cdot\left(\frac{x}{R}\right)^2}\right] \qquad x := \text{Maximize}\left(VR_{Shepard}, x\right)$$

$$\frac{x}{R} = 0.235 \qquad \frac{x}{x_{max}} = 47.088 \cdot \%$$

$$x\_R_{Shepard} := -1 + \sqrt{\frac{7}{5} + \frac{2}{5}\cdot m_r} = 0.235$$

Shepard's result differs from the distance-based result above (x/R = 0.235 vs. 0.174) because he neglected cue tip inefficiency and the drag distance of the ball.

Now, assuming a perfect tip, and neglecting the skid phase, the distance analysis yields the same result as the rolling-speed analysis (i.e., the optimal tip height for rolling distance is the same as that for rolling speed).

$$f(x) := d\left(v_s, x, \eta, m_r, \mu_s, \mu_r\right) \qquad x := \text{Maximize}(f, x) \qquad \frac{x}{R} = 0.235 \qquad \frac{x}{x_{max}} = 47.088 \cdot \%$$

Accounting for tip inefficiency and considering rolling only, the optimal tip height is:

$$\eta := .87 \qquad VR(x) := \frac{v_{skid}\left(v\left(v_s, x, \eta, m_r\right), \omega\left(v_s, x, \eta, m_r\right)\right)}{v_s}$$

$$x := 0 \qquad x := \text{Maximize}(VR, x) \qquad \frac{x}{R} = 0.186 \qquad \frac{x}{x_{max}} = 37.263 \cdot \%$$

Again, this differs from the result above (x/R = 0.186 vs. 0.174), due to the effects of drag distance.

And again, including the effects of both tip inefficiency and skid distance, the optimal tip height, under typical conditions, is about 17.5% of the ball radius or about 35% of the maximum tip offset:

$\mu_s := 0.2 \quad f(x) := d(v_s, x, \eta, m_r, \mu_s, \mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.174 \quad \frac{x}{x_{max}} = 34.864 \cdot \%$

Now let's look at the effects of different conditions, relative to typical values.

With a lighter (10% less weight) cue, the optimal tip height is a little higher than normal:

$f(x) := d\left(v_s, x, \eta, \frac{m_r}{.90}, \mu_s, \mu_r\right) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.184 \quad \frac{x}{x_{max}} = 36.708 \cdot \%$

And with a heavier (10% more weight) cue, the optimal tip height is a little lower than normal:

$f(x) := d\left(v_s, x, \eta, \frac{m_r}{1.10}, \mu_s, \mu_r\right) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.166 \quad \frac{x}{x_{max}} = 33.14 \cdot \%$

With slicker cloth (50% less sliding friction), the optimal tip height is a little lower than normal:

$f(x) := d(v_s, x, \eta, m_r, 0.5\mu_s, \mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.162 \quad \frac{x}{x_{max}} = 32.462 \cdot \%$

And with stickier cloth (50% more sliding friction), the optimal tip height is a little higher than normal:

$f(x) := d(v_s, x, \eta, m_r, 1.5\mu_s, \mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.178 \quad \frac{x}{x_{max}} = 35.665 \cdot \%$

With faster cloth (50% less rolling resistance), the optimal tip height is a little higher than normal:

$f(x) := d(v_s, x, \eta, m_r, \mu_s, 0.5\mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.18 \quad \frac{x}{x_{max}} = 36.065 \cdot \%$

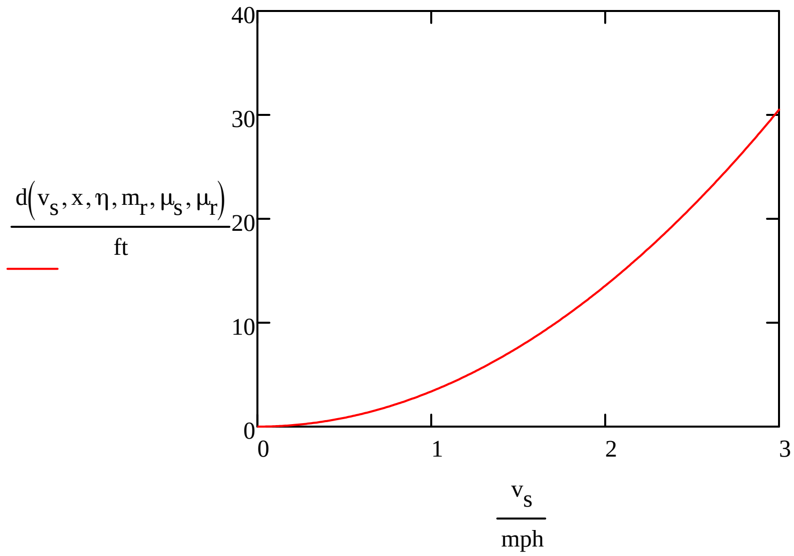
And with slower cloth (50% more rolling resistance), the optimal tip height is a little lower than normal:

$f(x) := d(v_s, x, \eta, m_r, \mu_s, 1.5\mu_r) \quad x := \text{Maximize}(f, x) \quad \frac{x}{R} = 0.168 \quad \frac{x}{x_{max}} = 33.662 \cdot \%$

BOTTOM LINE: When trying to control the CB travel distance very accurately and consistently, the best tip height to use is about 16-19% of the ball's radius above the center. The exact amount varies some with equipment and conditions.

As a final observation, note how ball travel distance (including both drag and rolling) varies with cue speed at the optimal tip offset:

$$x := 0.174 \cdot R \qquad v_s := 0, 0.01 \cdot mph .. 3 \cdot mph$$



As with just rolling, total travel distance vs. speed is a square relationship. As you double the cue speed, the distance quadruples.

$$\frac{d(2 \cdot mph, x, \eta, m_r, \mu_s, \mu_r)}{d(1 \cdot mph, x, \eta, m_r, \mu_s, \mu_r)} = 4$$