technical proof

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TP A.19
Masse shot aiming method, and curved cue ball paths

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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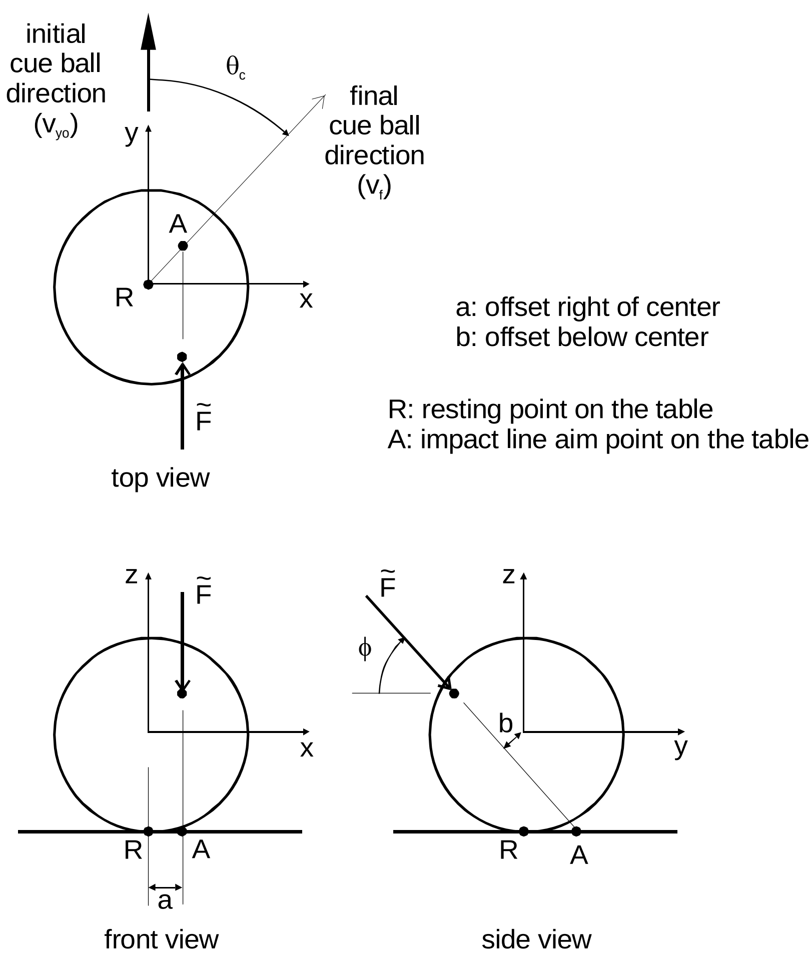
The cue ball trajectory is based on the equations derived in TP A.4. See TP A.4 for Equation number references and background information.



The cue ball is assumed to be struck in the y direction with the angled impulse as shown above. From linear impulse and momentum, the initial cue ball speed components, after cue stick impact, are:

$$v_{xo} = 0 \tag{1}$$

$$v_{yo} = \frac{\tilde{F}}{m}\cos(\phi) = v_e \cos(\phi) \tag{2}$$

where v_e is the effective speed of a straight-on impact.

From angular impulse and momentum, the spin components of the cue ball about the x and y axes, after cue stick impact, are related to the impulse according to:

$$\tilde{F}b = I\omega_{xo} = \left(\frac{2}{5}mR^2\right)\omega_{xo} \tag{3}$$

$$\tilde{F}a\sin(\phi) = I\omega_{yo} = \left(\frac{2}{5}mR^2\right)\omega_{yo} \tag{4}$$

Therefore, the initial spin components, using Equation 2, can be written as:

$$\omega_{xo} = \frac{5v_e}{2R^2}b \tag{5}$$

$$\omega_{yo} = \frac{5v_e}{2R^2}a\sin(\phi) \tag{6}$$

We can now find the final cue ball direction from Equation 23 in TP A.4:

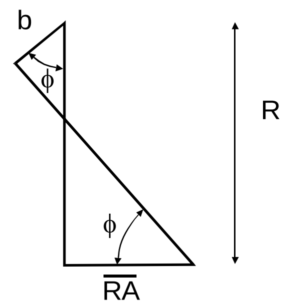
$$\tan(\theta_c) = \frac{5v_{xo} + 2R\omega_{yo}}{5v_{yo} - 2R\omega_{xo}} = \frac{0 + \frac{5v_e}{R}a\sin(\phi)}{5v_e\cos(\phi) - \frac{5v_e}{R}b} = \frac{a\sin(\phi)}{R\cos(\phi) - b} \tag{7}$$

As proved by Coriolis, the final cue ball direction can also be predicted from the geometry shown in the illustration above. From the top view, the final cue ball direction can be expressed as:

$$\tan(\theta_c) = \frac{a}{\overline{RA}} \tag{8}$$

From the side view in the illustration above, it can be shown (see the figure below) that:

$$R\cos(\phi) = b + \overline{RA}\sin(\phi) \tag{9}$$



Equation 9 can be rearranged to give:

$$\overline{RA} = \frac{R\cos(\phi)}{\sin(\phi)} \tag{10}$$

Using Equation 10 in Equation 8 results in the same expression as Equation 7. Therefore, the aiming method proposed by Coriolis is shown to be valid. Note that the friction between the ball and the table during cue stick impact is is not accounted for in this analysis, but Coriolis also showed that this friction has no effect on the final cue ball direction.

With typcial conditions, the final CB angle usually comes up a little short of the angle predicted by the Coriolis method. This could be due in part to squirt and multiple-body collision effects associated with jamming the cue ball between the tip and slate.

Now we can define the cue ball trajectory based on the analyses above and in TP A.4.

NOTE: All parameters are expressed in metric (SI) equivalent values for dimensionless analysis

ball properties:

$D := \frac{2.25 \cdot in}{m}$ $\qquad R := \frac{D}{2}$ $\qquad D = 0.057$

coefficient of friction between the cue ball and table cloth:

$\mu := 0.2$

gravity

$g := \frac{g}{\frac{m}{s^2}}$ $\qquad g = 9.807$

From Equation 1, 2, 5, and 6 above,

$$v_{xo} := 0$$

$$\omega_{xo}(v_e, b) := \frac{5 \cdot v_e}{2 \cdot R^2} \cdot b$$

$$v_{yo}(v_e, \phi) := v_e \cdot \cos(\phi)$$

$$\omega_{yo}(v_e, a, \phi) := \frac{5 \cdot v_e}{2 \cdot R^2} \cdot a \cdot \sin(\phi)$$

From Equation 14 in TP A.4,

$$v_{Cox}(v_e, a, b, \phi) := v_{xo} - R \cdot \omega_{yo}(v_e, a, \phi)$$

$$v_{Coy}(v_e, a, b, \phi) := v_{yo}(v_e, \phi) + R \cdot \omega_{xo}(v_e, b)$$

$$v_{Co}(v_e, a, b, \phi) := \sqrt{v_{Cox}(v_e, a, b, \phi)^2 + v_{Coy}(v_e, a, b, \phi)^2}$$

From Equation 20 in TP A.4,

$$\Delta t(v_e, a, b, \phi) := \frac{2 \cdot v_{Co}(v_e, a, b, \phi)}{7 \cdot \mu \cdot g}$$

From Equations 21 and 22 in TP A.4,

$$v_{xf}(v_e, a, b, \phi) := v_{xo} - \mu \cdot g \cdot \frac{v_{Cox}(v_e, a, b, \phi)}{v_{Co}(v_e, a, b, \phi)} \cdot \Delta t(v_e, a, b, \phi)$$

$$v_{yf}(v_e, a, b, \phi) := v_{yo}(v_e, \phi) - \mu \cdot g \cdot \frac{v_{Coy}(v_e, a, b, \phi)}{v_{Co}(v_e, a, b, \phi)} \cdot \Delta t(v_e, a, b, \phi)$$

From Equations 24 and 25 in TP A.4,

$$x_c(v_e, a, b, \phi, t) := v_{xo} \cdot t - \frac{\mu \cdot g \cdot v_{Cox}(v_e, a, b, \phi)}{2 \cdot v_{Co}(v_e, a, b, \phi)} \cdot t^2$$

$$y_c(v_e, a, b, \phi, t) := v_{yo}(v_e, \phi) \cdot t - \frac{\mu \cdot g \cdot v_{Coy}(v_e, a, b, \phi)}{2 \cdot v_{Co}(v_e, a, b, \phi)} \cdot t^2$$

From the analysis in TP A.4, the entire cue ball path is defined by:

$$x\left(v_e,a,b,\phi,t\right) := \left|\begin{array}{l} \Delta T \leftarrow \Delta t\left(v_e,a,b,\phi\right) \\ x_c\left(v_e,a,b,\phi,t\right) \ \text{ if } \ t \le \Delta T \\ \left[x_c\left(v_e,a,b,\phi,\Delta T\right) + v_{xf}\left(v_e,a,b,\phi\right)\cdot(t - \Delta T)\right] \ \text{ otherwise} \end{array}\right.$$

$$y\left(v_e,a,b,\phi,t\right) := \left|\begin{array}{l} \Delta T \leftarrow \Delta t\left(v_e,a,b,\phi\right) \\ y_c\left(v_e,a,b,\phi,t\right) \ \text{ if } \ t \le \Delta T \\ \left[y_c\left(v_e,a,b,\phi,\Delta T\right) + v_{yf}\left(v_e,a,b,\phi\right)\cdot(t - \Delta T)\right] \ \text{ otherwise} \end{array}\right.$$

From Equation 7 above, the final cue ball angle can be found with:

$$\theta_c(a,b,\phi) := \text{angle}(R\cdot\cos(\phi) - b, a\cdot\sin(\phi))$$

Parameters used in plots below:

$T := 5$	number of seconds to display
$t := 0, 0.01 .. T$	0.01 second plotting increment

Equation for the final direction:

$$x_f(t) := \frac{t}{T}\cdot 2$$

$$y_f(a,b,\phi,t) := \frac{x_f(t)}{\tan\left(\theta_c(a,b,\phi)\right)}$$

Equation for the ball (for scale)

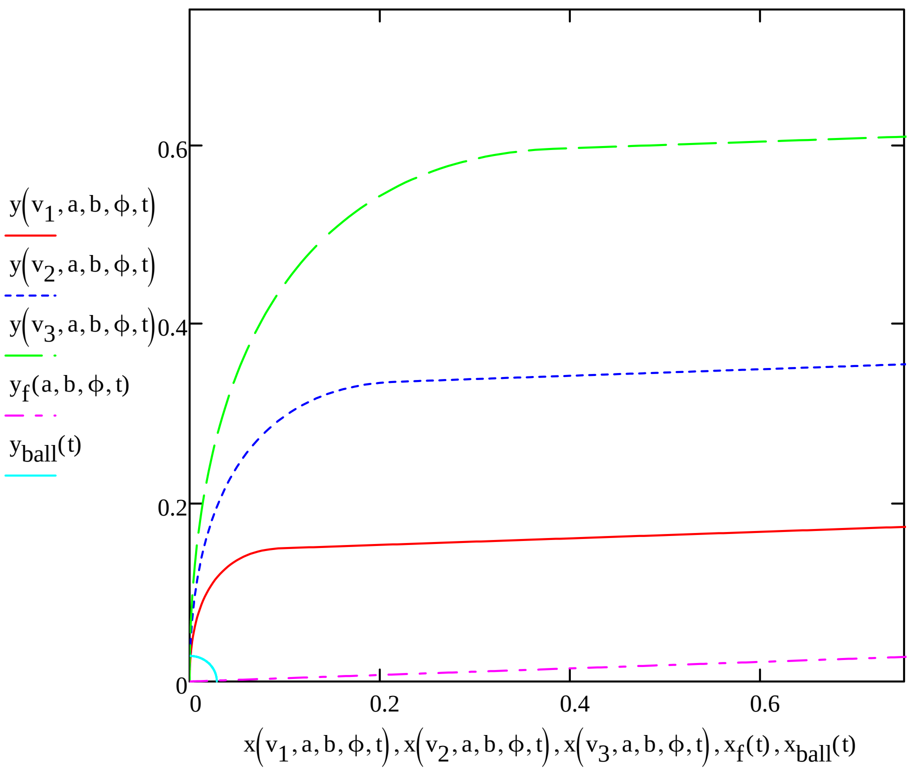
$$x_{ball}(t) := R\cdot\cos\left(\frac{t}{T}\cdot\frac{\pi}{2}\right)$$

$$y_{ball}(t) := R\cdot\sin\left(\frac{t}{T}\cdot\frac{\pi}{2}\right)$$

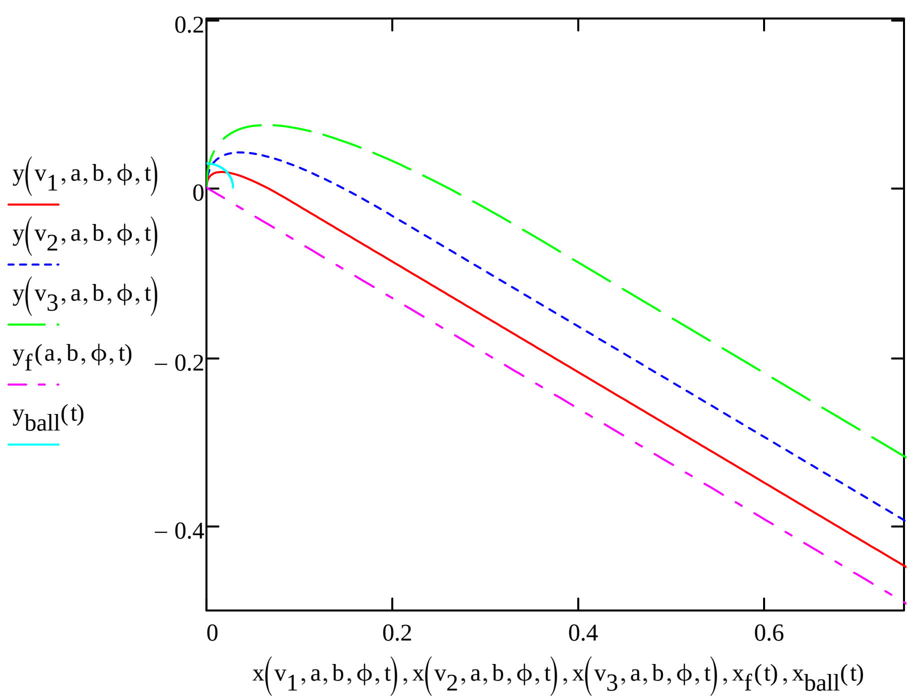
various effective impact speeds (in mph, converted to m/s):

$$v_1 := \frac{6\cdot\text{mph}}{\frac{m}{s}} \qquad v_2 := \frac{9\cdot\text{mph}}{\frac{m}{s}} \qquad v_3 := \frac{12\cdot\text{mph}}{\frac{m}{s}}$$

$\phi := 75\cdot deg \qquad a := 0.25\cdot R \qquad b := 0.25\cdot R$



$\phi := 85\cdot deg \qquad a := 0.25\cdot R \qquad b := 0.25\cdot R$



Comparison of draw and follow shot swerve for a medium-fast-speed shot under typical ball and table conditions:

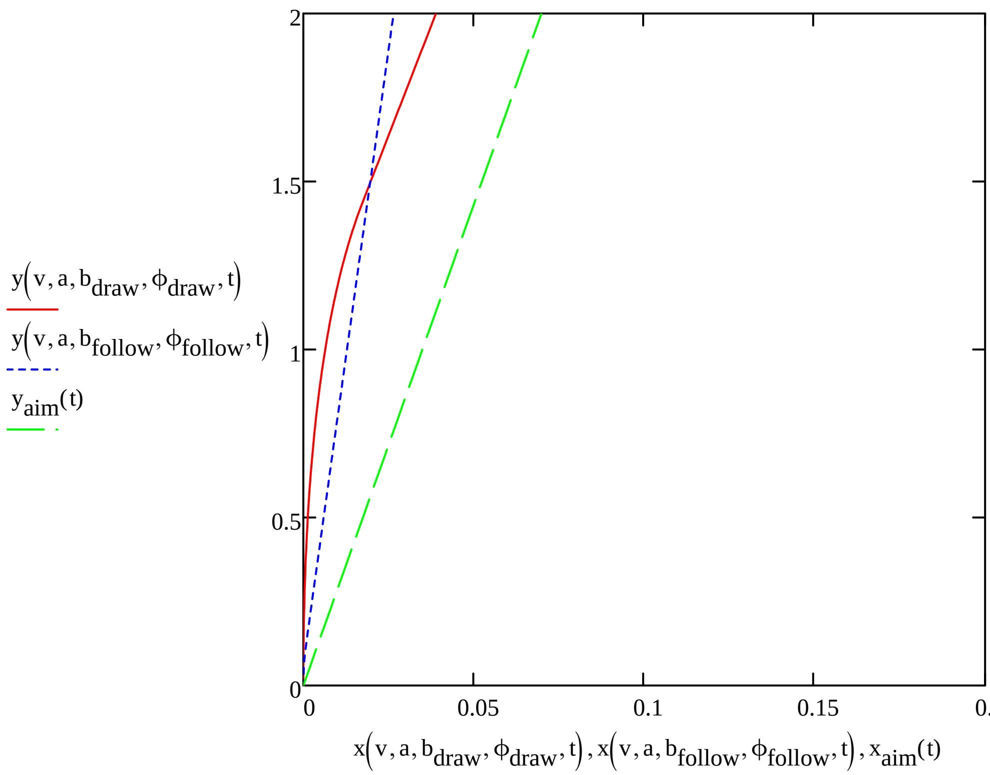
$v := \dfrac{6 \cdot mph}{\dfrac{m}{s}}$ $\quad a := 0.5 \cdot R \cdot \cos(45 \cdot deg)$ $\quad$ same amount of English for both: 1:30pm (follow) or 4:30pm (draw)

$b_{draw} := a \qquad b_{follow} := -a$ $\quad$ same amount of top or bottom spin for both

$\phi_{draw} := 4 \cdot deg \qquad \phi_{follow} := 3 \cdot deg$ $\quad$ slightly higher cue elevation for the draw shot, but typical values

initial aiming line with a typical amount of squirt (2 degrees):

$$x_{aim}(t) := \frac{t}{T} \qquad y_{aim}(t) := \frac{x_{aim}(t)}{\tan(2 \cdot deg)}$$



The draw shot exhibits more "effective squirt" (the net effect of both squirt and swerve, AKA squerve) over the shorter shot distances (0-4.5 ft); but, over longer shot distances (>4.5 ft), the follow shot exhibits more "effective squirt." The crossover distance (4.5 ft) depends on shot speed and ball/table conditions.

Here are the final cue-ball trajectory angles after sliding ceases and rolling begins:

$\theta_c(a, b_{draw}, \phi_{draw}) = 2.193\ deg \qquad \theta_c(a, b_{follow}, \phi_{follow}) = 0.784\ deg$

Final cue ball angle for various cue stick elevations:

$a := 0.25 \cdot R$ $\qquad$ $b := 0.25 \cdot R$ $\qquad$ $\phi := 0 \cdot deg, 1 \cdot deg .. 90 \cdot deg$