technical proof

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TP B.10
Draw shot cue elevation effects

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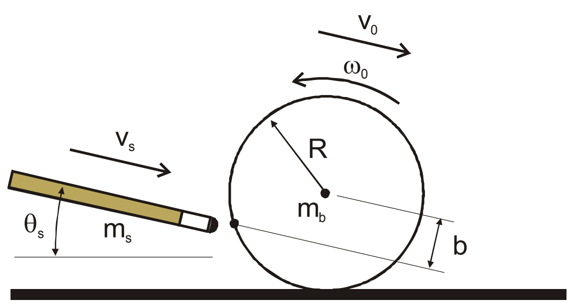
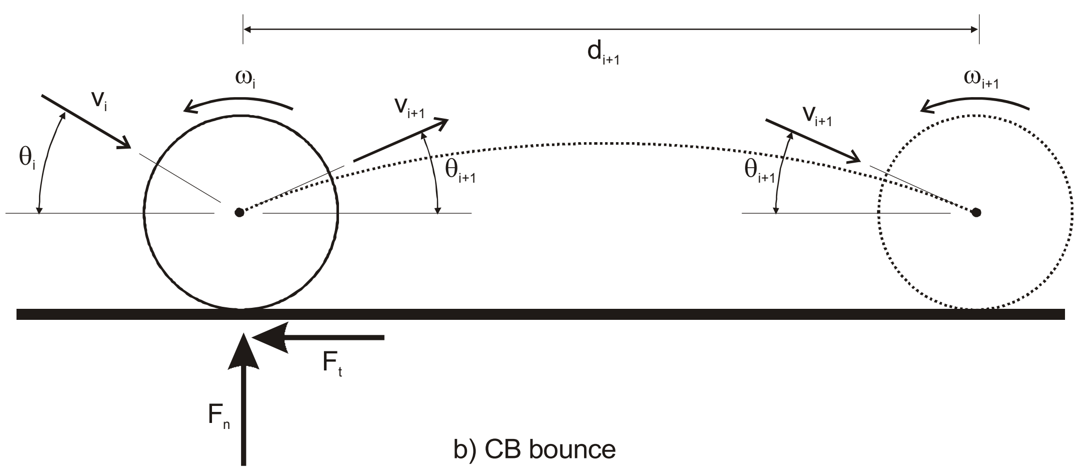


a) cue stick impact



b) CB bounce

The data and equations on this page are from TP B.9.

Relevant physical parameters:

$R := 1.125 \cdot in$ ball radius

$b_{max} := \frac{R}{2}$ generally accepted miscue limit

$\mu_s := 0.2$ typical ball-cloth coefficient of sliding friction

$m_r := \frac{6}{19}$ typical ball-mass-to-cue-mass ratio (m_b/m_s):

$\eta := 0.87$ typical cue tip efficiency

$e_t := 0.5$ typical ball-table coefficient of restitution

$rps := \frac{rpm \cdot min}{sec}$ revs per second

$v_s := 12 \cdot mph$ typical fast cue speed

Speed and spin loss, and spin ratio, for a sliding CB (e.g., with a level-cue draw shot):

$$v_{drag}(v, x) := \sqrt{v^2 - 2 \cdot \mu_s \cdot g \cdot x} \qquad \omega_{drag}(v, \omega, x) := \omega - \frac{5}{2 \cdot R} \cdot \left(v - \sqrt{v^2 - 2 \cdot \mu_s \cdot g \cdot x}\right)$$

$$drag\_ratio(v, \omega, x) := \frac{\omega_{drag}(v, \omega, x) \cdot R}{v_{drag}(v, x)}$$

CB speed and spin from cue speed (v_s) and tip offset (b):

$$v_{CB}(v_s, b) := v_s \cdot \frac{1 + \sqrt{\eta - \frac{1 - \eta}{m_r} \cdot \left[1 + \frac{5}{2} \cdot \left(\frac{b}{R}\right)^2\right]}}{\left[1 + m_r + \frac{5}{2} \cdot \left(\frac{b}{R}\right)^2\right]} \qquad \omega_{CB}(v_s, b) := \frac{5}{2} \cdot v_{CB}(v_s, b) \cdot \frac{b}{R^2}$$

Typical fast cue speed with near-maximum tip offset:

$v_s := 15 \cdot mph \qquad b := b_{max} = 0.563 \cdot in$

Cue ball (CB) speed, spin, and angle after tip impact:

$v_0 := v_{CB}(v_s, b) = 11.194 \cdot mph \qquad \omega_0 := \omega_{CB}(v_s, b) = 34.841 \cdot rps$

NOTE - I will assume the cue tip delivers most of its impulse to the CB before any significant force build up between the CB and the table (due to cloth compression and rebound off the slate). This assumption is supported by HSV B.44 for modest cue elevations.

On each bounce of the CB, including the first bounce after tip impact, the ball speed, spin, and angle change according to impact-momentum principles. See the illustrations on the 1st page for the terminology used.

The normal and tangential components of the incoming velocity are:

$$vn_i = v_i \cdot \sin(\theta_i) \qquad\qquad vt_i = v_i \cdot \cos(\theta_i)$$

After the bounce, the normal component is reduced according to the coefficient of restitution between the ball and table:

$$vn_{i+1} = e_t \cdot vn_i = e_t \cdot v_i \cdot \sin(\theta_i)$$

The normal impulse corresponding to this momentum change is:

$$F'_n = m_b \cdot (vn_i + vn_{i+1}) = m_b \cdot (1 + e_t) \cdot \sin(\theta_i) \cdot v_i$$

Assuming the ball still has backspin after rebound (i.e., $\omega_{i+1}>0$), the tangential impulse will be:

$$F'_t = \mu_s \cdot F'_n = \mu_s \cdot m_b \cdot (1 + e_t) \cdot \sin(\theta_i) \cdot v_i$$

This changes the tangential momentum, resulting in a post-rebound tangential speed of:

$$vt_{i+1} = vt_i - \frac{F'_t}{m_b} = v_i \cdot \left[\cos(\theta_i) - \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i)\right]$$

The total post-rebound speed is:

$$v_{i+1} = \sqrt{(vn_{i+1})^2 + (vt_{i+1})^2} = v_i \sqrt{\left(e_t \cdot \sin(\theta_i)\right)^2 + \left[\cos(\theta_i) - \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i)\right]^2}$$

The tangential impulse also changes the angular momentum, resulting in a post-rebound spin of:

$$\omega_{i+1} = \omega_i - \frac{F'_t \cdot R}{I} = \omega_i - \frac{F'_t \cdot R}{\frac{2}{5} \cdot m_b \cdot R^2} = \omega_i - \frac{5}{2 \cdot R} \cdot \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i) \cdot v_i$$

The trajectory angle after the bounce is:

$$\theta_{i+1} = \mathrm{atan}\left(\frac{vn_{i+1}}{vt_{i+1}}\right) = \mathrm{atan}\left[\frac{e_t \cdot \sin(\theta_i)}{\cos(\theta_i) - \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i)}\right] = \mathrm{atan}\left[\frac{e_t}{\cot(\theta_i) - \mu_s \cdot (1 + e_t)}\right]$$

From basic projectile motion, the distance covered in the air after the bounce is:

$$d_{i+1} = \frac{(v_{i+1})^2}{g} \cdot \sin(2 \cdot \theta_{i+1})$$

Here's a program to calculate the CB spin and spin ratio at OB contact for a given cue elevation and drag distance:

```
elev_draw_shot(θs, ddrag) :=  | dtotal ← 0
                              | θ0 ← θs
                              | i ← 0
                              | while (dtotal < ddrag)
                              |   | "the CB hasn't reached the OB yet"
                              |   | if (θi = 0)
                              |   |   | "drag with no more bounces"
                              |   |   | x ← ddrag − dtotal
                              |   |   | d(i+1) ← x
                              |   |   | v(i+1) ← vdrag(vi, x)
                              |   |   | ω(i+1) ← ωdrag(vi, ωi, x)
                              |   |   | θ(i+1) ← 0
                              |   |   | i ← i + 1
                              |   |   | break
                              |   | "otherwise, the CB will bounce"
                              |   | vn(i+1) ← et·vi·sin(θi)
                              |   | vt(i+1) ← vi·[cos(θi) − μs·(1 + et)·sin(θi)]
                              |   | v(i+1) ← sqrt((vn(i+1))^2 + (vt(i+1))^2)
                              |   | ω(i+1) ← ωi − 5/(2·R)·μs·(1 + et)·sin(θi)·vi
                              |   | if [(v(i+1) < 0) ∨ (ω(i+1) < 0)]
                              |   |   | "CB doesn't make it to the OB with draw"
                              |   |   | v(i+1) ← 0
                              |   |   | ω(i+1) ← 0
                              |   |   | i ← i + 1
                              |   |   | break
                              |   | θ(i+1) ← atan[et / (cot(θi) − μs·(1 + et))]
                              |   | d(i+1) ← (v(i+1))^2 / g · sin(2·θ(i+1))
                              |   | dtotal ← dtotal + d(i+1)
```

Equations in the program:

$$vn_{i+1} \leftarrow e_t \cdot v_i \cdot \sin(\theta_i)$$

$$vt_{i+1} \leftarrow v_i \cdot \left[\cos(\theta_i) - \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i)\right]$$

$$v_{i+1} \leftarrow \sqrt{\left(vn_{i+1}\right)^2 + \left(vt_{i+1}\right)^2}$$

$$\omega_{i+1} \leftarrow \omega_i - \frac{5}{2 \cdot R} \cdot \mu_s \cdot (1 + e_t) \cdot \sin(\theta_i) \cdot v_i$$

$$\theta_{i+1} \leftarrow \operatorname{atan}\left[\frac{e_t}{\cot(\theta_i) - \mu_s \cdot (1 + e_t)}\right]$$

$$d_{i+1} \leftarrow \frac{\left(v_{i+1}\right)^2}{g} \cdot \sin(2 \cdot \theta_{i+1})$$

$$d_{total} \leftarrow d_{total} + d_{i+1}$$

$$\begin{aligned}
&\quad \text{if } \left[\left(d_{i+1} < 0.1\cdot in\right) \wedge \left(\theta_{i+1} < 1\cdot deg\right)\right] \\
&\quad\quad \text{"assume the bouncing ceases"} \\
&\quad\quad \theta_{i+1} \leftarrow 0 \\
&\quad i \leftarrow i + 1 \\
&spin \leftarrow \omega_i \\
&ratio \leftarrow \frac{\omega_i \cdot R}{v_i \cdot \cos\left(\theta_i\right)} \\
&\text{"return all calculated data"} \\
&\left(i \quad \frac{d}{ft} \quad \frac{v}{mph} \quad \frac{\omega}{rps} \quad \frac{\theta}{deg} \quad \frac{spin}{rps} \quad ratio\right)^T
\end{aligned}$$

Functions to extract spin and spin-to-speed ratio from the program results:

$spin\left(\theta_s, d_{drag}\right) := elev\_draw\_shot\left(\theta_s, d_{drag}\right)_5$ $\qquad ratio\left(\theta_s, d_{drag}\right) := elev\_draw\_shot\left(\theta_s, d_{drag}\right)_6$

For a level cue, the program results are the same as for a drag shot:

$\theta_s := 0\cdot deg \qquad d_{drag} := 2\cdot ft \qquad spin\left(\theta_s, d_{drag}\right) = 31.43 \qquad ratio\left(\theta_s, d_{drag}\right) = 1.186$

$\omega_{drag}\left(v_0, \omega_0, d_{drag}\right) = 31.43\cdot rps \qquad drag\_ratio\left(v_0, \omega_0, d_{drag}\right) = 1.186$

Here are some example data at a typical cue elevation and medium drag distance:

$\theta_s := 5\cdot deg \qquad d_{drag} := 2\cdot ft \qquad results := elev\_draw\_shot\left(\theta_s, d_{drag}\right)$

$N := results_0 = 9$ number of bounces

$db := results_1$ distances between bounces (in feet)

$db_1 = 0.708 \quad db_2 = 0.349 \quad db_3 = 0.174 \quad db_4 = 0.086 \quad db_5 = 0.043$

$vb := results_2$ ball speed after each bounce (in mph)

$vb_1 = 10.87 \quad vb_2 = 10.715 \quad vb_3 = 10.64 \quad vb_4 = 10.603 \quad vb_5 = 10.585$

$\omega b := results_3$ ball spin after each bounce (in rps)

$\omega b_1 = 33.019 \quad \omega b_2 = 32.108 \quad \omega b_3 = 31.652 \quad \omega b_4 = 31.424 \quad \omega b_5 = 31.311$

$\theta b := results_4$ ball bounce angle (in deg)

$\theta b_1 = 2.572 \quad \theta b_2 = 1.304 \quad \theta b_3 = 0.657 \quad \theta b_4 = 0.33 \quad \theta b_5 = 0.165$

$results_5 = 30.143 \qquad spin\left(\theta_s, d_{drag}\right) = 30.143$

$results_6 = 1.164 \qquad ratio\left(\theta_s, d_{drag}\right) = 1.164$

Example data at higher cue elevation:

$\theta_s := 15\cdot deg \quad d_{drag} := 2\cdot ft \qquad results := elev\_draw\_shot(\theta_s, d_{drag})$

$N := results_0 = 2$ number of bounces

$db := results_1$ distances between bounces (in feet)

$db_1 = 1.926 \quad db_2 = 0.921$

$vb := results_2$ ball speed after each bounce (in mph)

$vb_1 = 10.049 \quad vb_2 = 9.537$

$\omega b := results_3$ ball spin after each bounce (in rps)

$\omega b_1 = 29.43 \quad \omega b_2 = 26.725$

$\theta b := results_4$ ball bounce angle (in deg)

$\theta b_1 = 8.289 \quad \theta b_2 = 4.356$

$results_5 = 26.725 \qquad spin(\theta_s, d_{drag}) = 26.725$

$results_6 = 1.129 \qquad ratio(\theta_s, d_{drag}) = 1.129$

Comparing elevated before-bounce and after-bounce spin and ratio to a level-cue drag shot:

original CB speed, spin, and spin ratio:

$$v_0 = 11.194\cdot mph \qquad \omega_0 = 34.841\cdot rps \qquad \frac{\omega_0\cdot R}{v_0} = 1.25$$

just before the second bounce: $d_{drag} := 1.925\cdot ft$

level cue: $\omega_{drag}(v_0, \omega_0, d_{drag}) = 31.561\cdot rps \qquad \dfrac{\omega_{drag}(v_0, \omega_0, d_{drag})\cdot R}{v_{drag}(v_0, d_{drag})} = 1.188$

elevated cue: $spin(\theta_s, d_{drag}) = 29.43 \qquad ratio(\theta_s, d_{drag}) = 1.189$

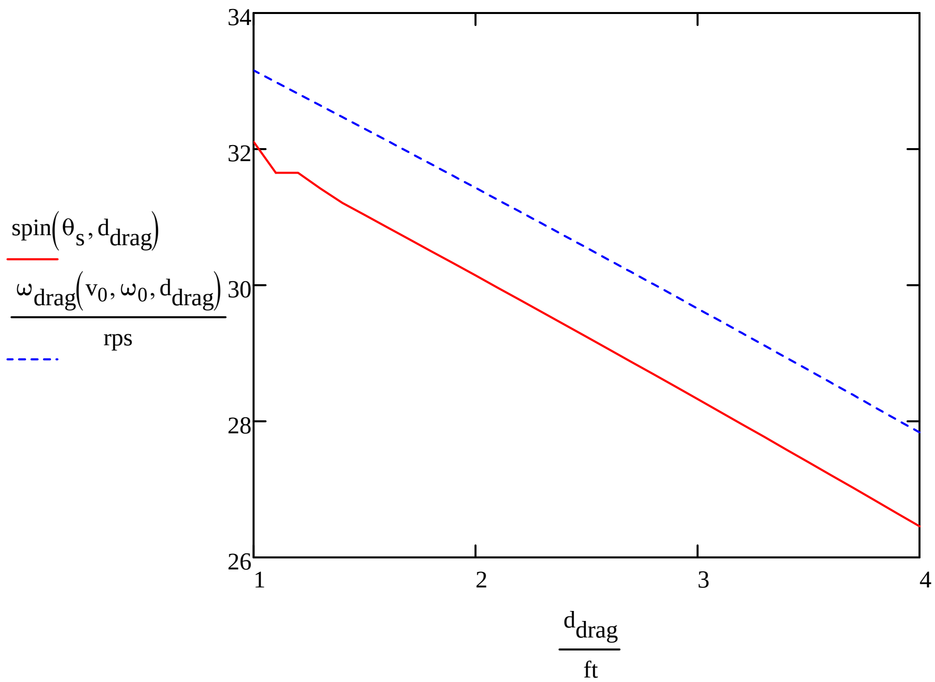
just after the second bounce: $d_{drag} := 1.927\cdot ft$

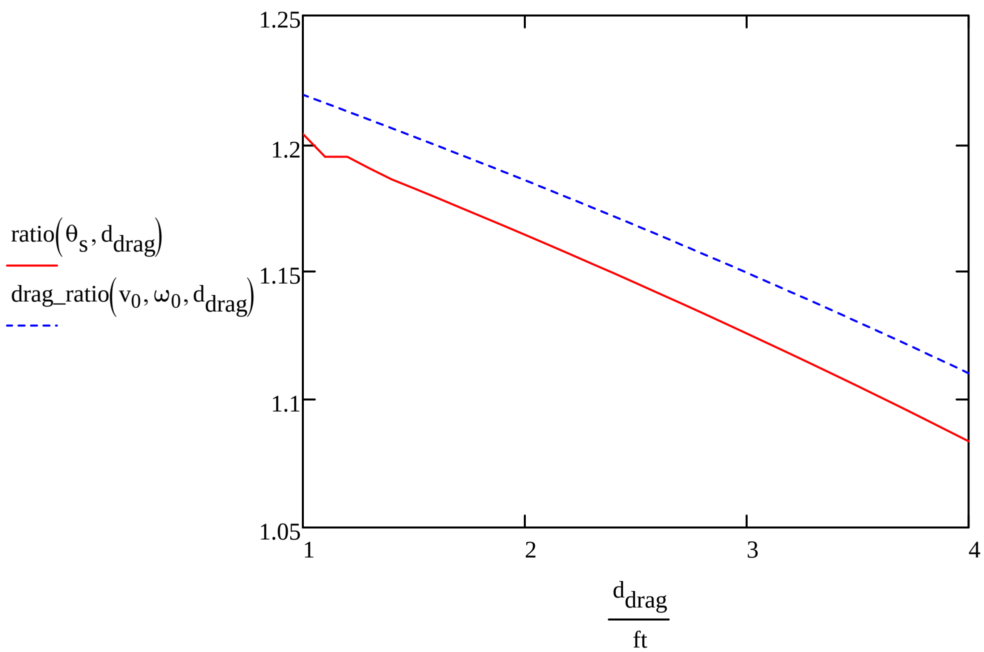
level cue: $\omega_{drag}(v_0, \omega_0, d_{drag}) = 31.558\cdot rps \qquad \dfrac{\omega_{drag}(v_0, \omega_0, d_{drag})\cdot R}{v_{drag}(v_0, d_{drag})} = 1.188$

elevated cue: $spin(\theta_s, d_{drag}) = 26.725 \qquad ratio(\theta_s, d_{drag}) = 1.129$

Now let's look at how spin and spin ratio vary with drag distance for a typical cue elevation, as compared to a level-cue drag shot:

$\theta_s := 5 \cdot deg$ $\qquad$ $d_{drag} := 1 \cdot ft, 1.1 \cdot ft .. 4 \cdot ft$

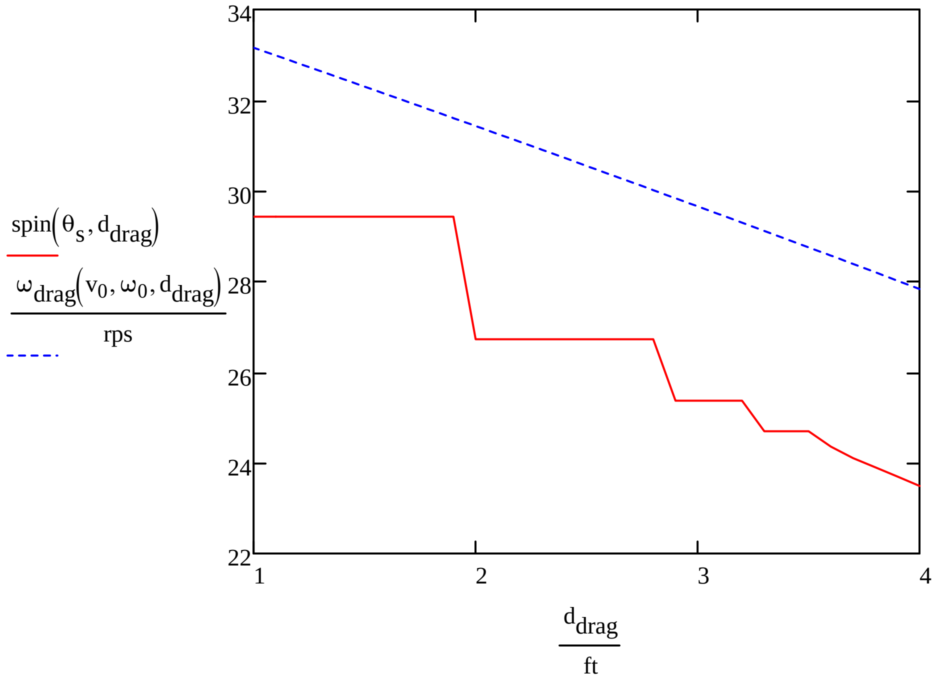


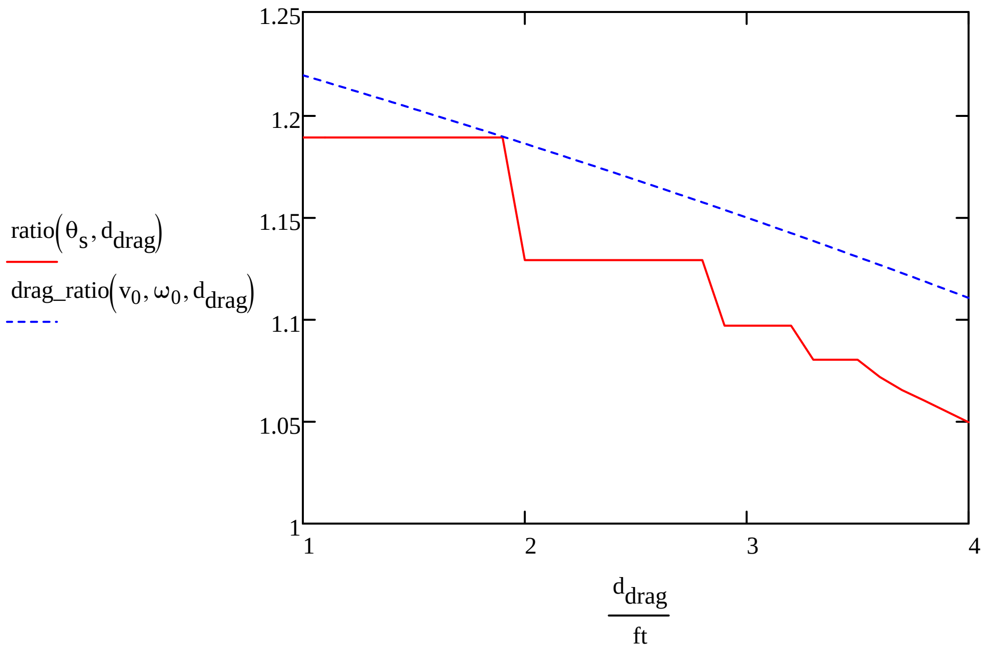


With a near level cue, there is little effect from the CB bouncing, and the CB doesn't bounce for very long (see the small spikes in the curves on the left side of the graphs).

Now let's look at how spin and spin ratio vary with drag distance for an elevated cue, as compared to a level-cue drag shot:

$\theta_s := 15 \cdot deg$ $\qquad$ $d_{drag} := 1 \cdot ft, 1.1 \cdot ft .. 4 \cdot ft$

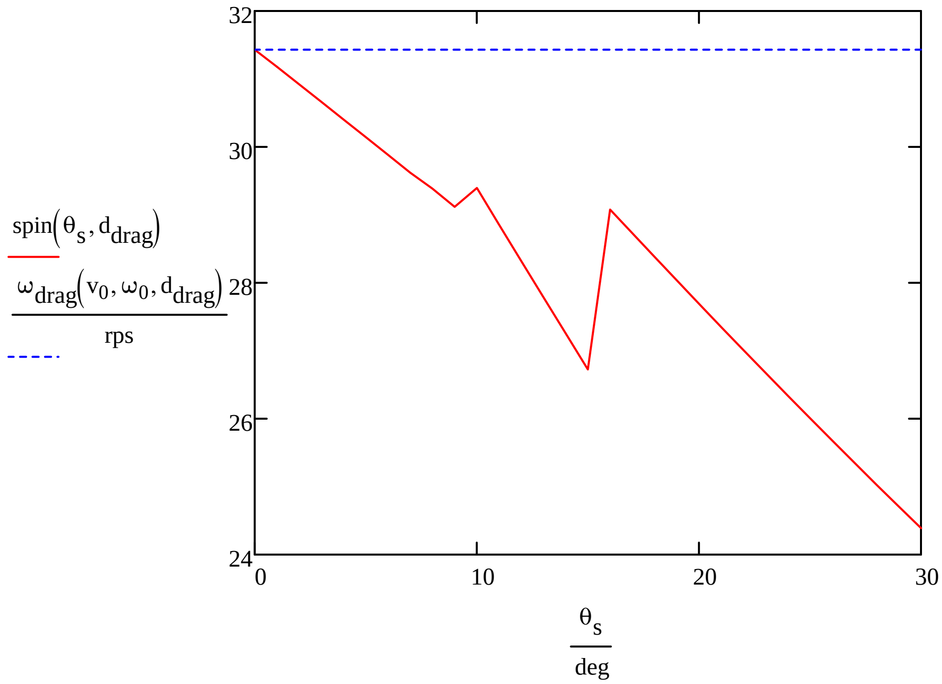


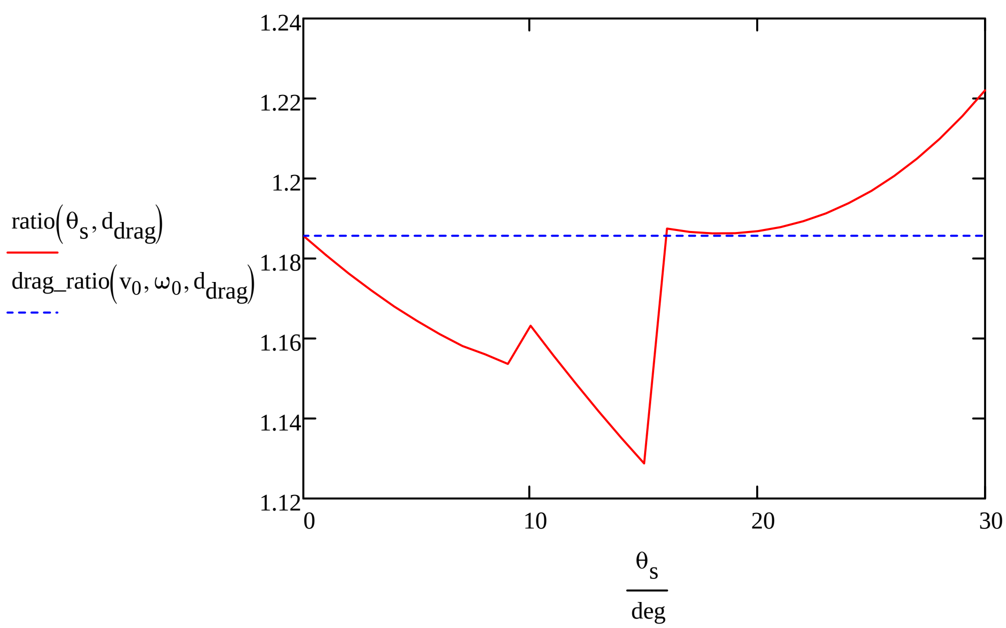


Notice how both the spin and the spin ratio take "hits" after each bounce, but don't change while the CB is in airborne between the bounces.

Now let's look at how spin and spin ratio vary with cue elevation, as compared to a level-cue drag shot:

$$\theta_s := 0\cdot deg, 1\cdot deg .. 30\cdot deg \qquad d_{drag} := 2\cdot ft$$





The reasons for the spikes in the graphs above are related to the distances at which bounces occur for different cue elevations. The following data explain the two spikes in the curves:

$\theta_s := 9\cdot deg$ results := elev_draw_shot(θ_s, d_{drag}) spin$(\theta_s, d_{drag}) = 29.118$

$N := results_0 = 3$ $d := results_1$ $d_1 + d_2 = 1.834$ ratio$(\theta_s, d_{drag}) = 1.154$

$\theta_s := 10\cdot deg$ results := elev_draw_shot(θ_s, d_{drag}) spin$(\theta_s, d_{drag}) = 29.396$

$N := results_0 = 2$ $d := results_1$ $d_1 + d_2 = 2.016$ ratio$(\theta_s, d_{drag}) = 1.163$

$\theta_s := 15\cdot deg$ results := elev_draw_shot(θ_s, d_{drag}) spin$(\theta_s, d_{drag}) = 26.725$

$N := results_0 = 2$ $d := results_1$ $d_1 = 1.926$ ratio$(\theta_s, d_{drag}) = 1.129$

$\theta_s := 16\cdot deg$ results := elev_draw_shot(θ_s, d_{drag}) spin$(\theta_s, d_{drag}) = 29.079$

$N := results_0 = 1$ $d := results_1$ $d_1 = 2.029$ ratio$(\theta_s, d_{drag}) = 1.187$

Conclusions from all of the analysis and graphs above:

1. Elevating the cue reduces the amount of CB spin at OB contact, resulting in less draw distance (see TP B.8 for more info). The loss in spin is small for small cue elevations, but increases with more elevation (for a given cue speed and tip offset).

2. Modest cue elevations (about 0-15 degrees) reduce the spin-to-speed ratio of the CB at OB contact, resulting in "slower" draw (see TP B.9 for more info).

3. As you increase cue elevation above about 20 degrees, the spin-to-forward-speed ratio increases, allowing for "quicker" draw (see TP B.9 for more info). An extreme example is a highly elevated masse draw (pique) shot, where you create lots of backspin with very little forward speed.

Sometimes cue elevation is required to clear over an obstacle ball, or to prevent a double hit when there is a small gap between the CB and OB. And as noted above, with larger cue elevations, better "quick draw" action can result. However, **for maximum draw distance, a level cue (or as close to level is possible) appears to be best.**