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TP B.13
Rolling CB Deflection Angle Approximations

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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Relevant physical constants and parameters

$e := 0.95$ typical coefficient of restitution between balls

$D := \frac{2.25 \cdot in}{m}$ $R := \frac{D}{2}$ ball dimensions, converter to meters

$v_{slow} := \frac{3 \cdot mph}{\frac{m}{s}}$ typical slow CB speed, in meters/sec

$v_{fast} := \frac{7 \cdot mph}{\frac{m}{s}}$ typical medium-fast CB speed, in meters/sec

$\varphi := 0 \cdot deg, 1 \cdot deg .. 90 \cdot deg$ cut angle range

$f := 0, 0.01 .. 1$ ball-hit fraction range

From TP A.23, cut angle (ϕ) and ball-hit fraction (f) are related according to:

$$\varphi_f(f) := \text{asin}(1 - f) \qquad f_\varphi(\varphi) := 1 - \sin(\varphi)$$

From TP A.4, the ideal deflection angle for a rolling CB, neglecting ball inelasticity and friction is:

$$\theta_{ideal}(\varphi) := \text{atan}\left(\frac{\sin(\varphi) \cdot \cos(\varphi)}{\sin(\varphi)^2 + \frac{2}{5}}\right)$$

A very crude approximation for CB direction is called "back-of-the-ball" aiming, where you visualize the mirror image of the CB's ghost-ball position on the back of the OB, as if the OB were striking the mirror image of the CB. This approximation predicts that the CB deflection angle is the same as the cut angle of the shot:

$$\theta_{back}(\varphi) := \varphi$$

This approximation, which works fairly well close to a 1/2-ball hit, will be compared to others below.

As suggested by Onoda (Am. J. Phys., v57, n5, May, 1989), because of the shape of the curves in the plots below, the CB deflection angle can be approximated fairly well over certain ranges of shots. For thin hits (less than a 1/4-ball hit), the deflection angle is about 70% of the angle to the tangent line, for thick hits (more than a 3/4-ball hit), the deflection angle is about 2.5x the cut angle, and for cuts over the wide range from a 1/4-ball to a 3/4-ball hit, the deflection angle is fairly close to 30 degrees (i.e., the "natural angle"), which is the basis for the 30-degree rule. Equations for each of these cases are summarized below, and the approximation lines appear on the plots.

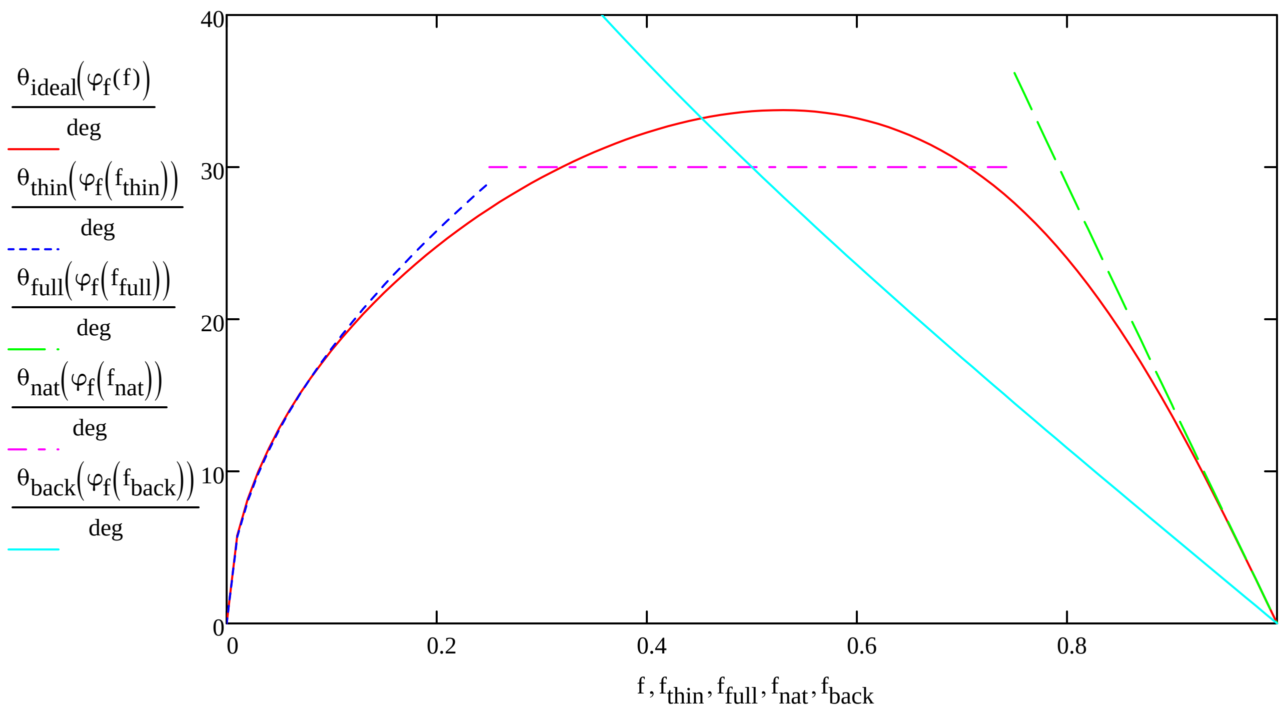
$\theta_{thin}(\varphi) := 0.7(90\cdot deg - \varphi)$ $\quad f_{thin} := 0, 0.01 .. 0.25$ $\quad \varphi_{thin} := \varphi_f\left(\frac{1}{4}\right), \varphi_f\left(\frac{1}{4}\right) + 0.01\cdot deg .. 90\cdot deg$

$\theta_{full}(\varphi) := 2.5\cdot\varphi$ $\quad f_{full} := 0.75, 0.76 .. 1$ $\quad \varphi_{full} := 0\cdot deg, 0.01\cdot deg .. \varphi_f\left(\frac{3}{4}\right)$

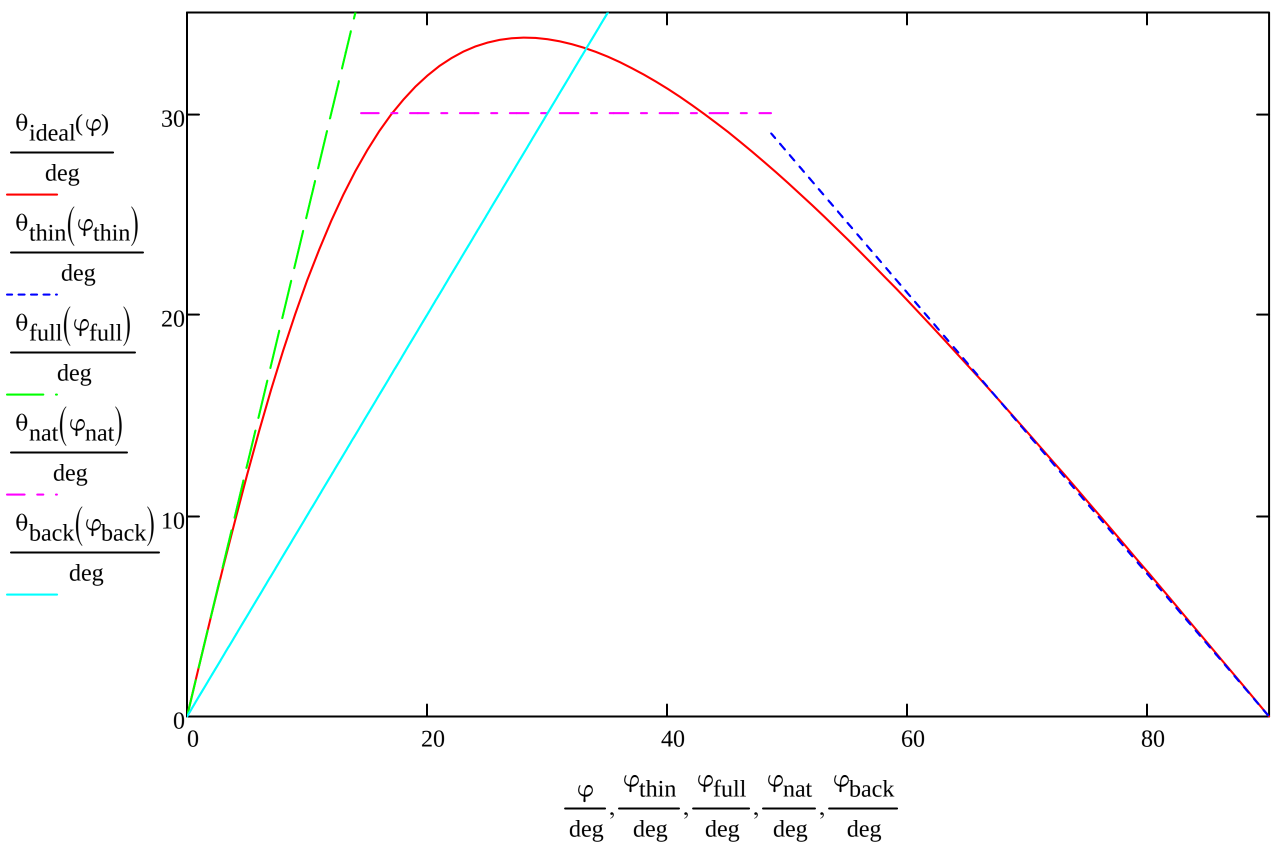
$\theta_{nat}(\varphi) := 30\cdot deg$ $\quad f_{nat} := 0.25, 0.26 .. 0.75$ $\quad \varphi_{nat} := \varphi_f\left(\frac{3}{4}\right), \varphi_f\left(\frac{3}{4}\right) + 0.01\cdot deg .. \varphi_f\left(\frac{1}{4}\right)$

$f_{back} := 0.35, 0.36 .. 1$ $\quad \varphi_{back} := 0\cdot deg, 0.01 deg .. \varphi_f(0.35)$

ideal CB deflection angle vs. ball-hit fraction:



ideal CB deflection angle vs. cut angle:



The plots and approximations above apply only to the idealized case of perfectly elastic balls with absolutely no friction between them. The following analysis takes into account both ball inelasticity and friction.

From TP A.14, the relative sliding speed between the CB and OB, at impact, is:

$$v_{rel} = \sqrt{\left(v\sin(\phi) - R\omega_z\right)^2 + \left(R\omega_x\cos(\phi)\right)^2}$$

For a rolling CB (ω_x = -v/R) with no English (ω_z=0), the relative sliding speed is the same as the CB speed for all cut angles:

$$v_{rel} = v$$

Therefore, from TP A.14, for a rolling CB with no English, the coefficient of sliding friction between the balls will vary with CB speed (in units of m/s) according to:

$$\mu(v) := 9.951 \times 10^{-3} + 0.108 \cdot e^{-1.088 \cdot v}$$

From TP A.6, the post-impact velocity components and final CB deflection angle for a rolling CB, accounting for ball inelasticity and friction, are given by:

$$\omega_{x0}(v,\phi) := \frac{v}{R}\cdot\left[\frac{5}{4}\cdot\mu(v)\cdot(1+e)\cdot\cos(\phi)^3 - 1\right]$$

$$\omega_{y0}(v,\phi) := \frac{v}{R}\cdot\left[\frac{5}{4}\cdot\mu(v)\cdot(1+e)\cdot\sin(\phi)\cdot\cos(\phi)^2\right]$$

$$v_{x0}(v,\phi) := \frac{v}{2}\cdot\sin(\phi)\cdot\cos(\phi)\cdot[1+e-\mu(v)\cdot(1+e)\cdot\cos(\phi)]$$

$$v_{y0}(v,\phi) := \frac{v}{2}\cdot\left[\sin(\phi)^2\cdot[2-\mu(v)\cdot(1+e)\cdot\cos(\phi)] + (1-e)\cdot\cos(\phi)^2\right]$$

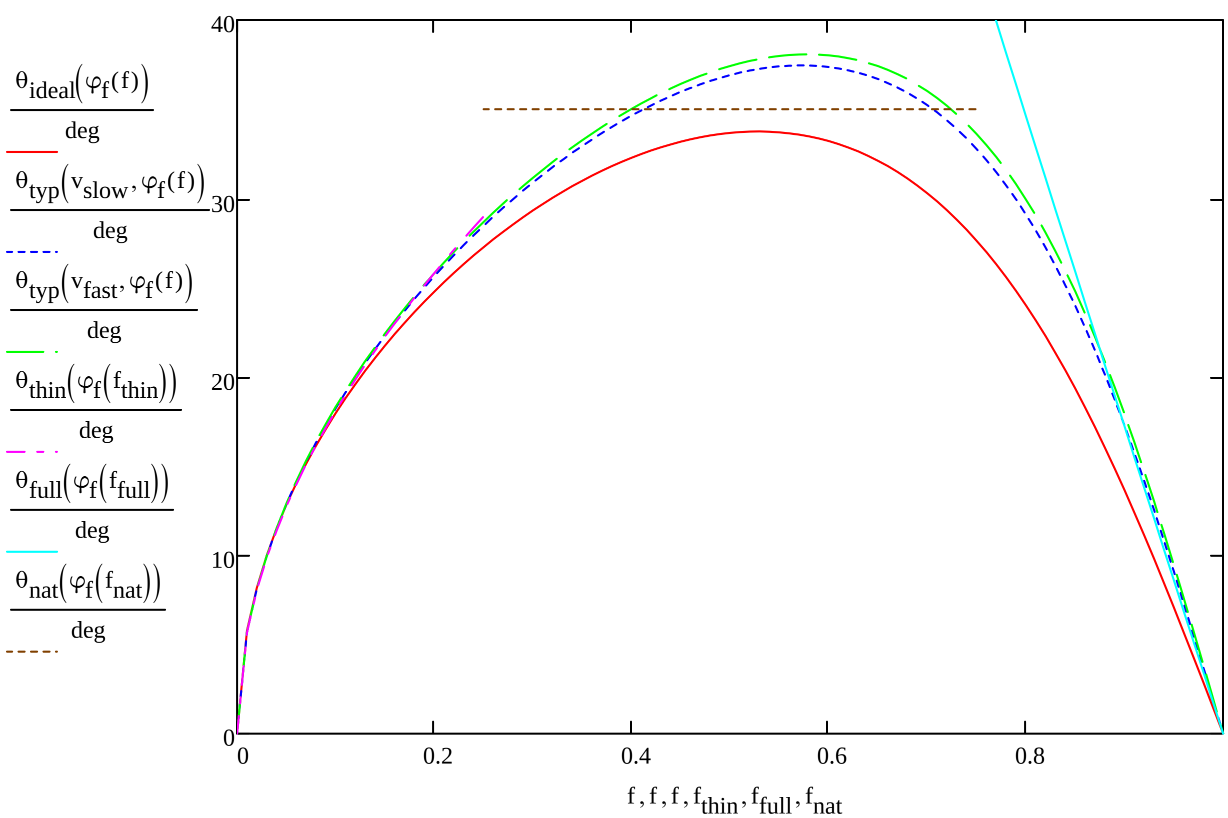
$$\theta_{typ}(v,\phi) := \operatorname{atan}\left(\frac{5\cdot v_{x0}(v,\phi) + 2\cdot R\cdot\omega_{y0}(v,\phi)}{5\cdot v_{y0}(v,\phi) - 2\cdot R\cdot\omega_{x0}(v,\phi)}\right)$$

Improved approximations that better match what happens with typical conditions on a real pool table are included on the plots below. The thick-hit angle is closer to 3x the cut angle, and the natural angle is closer to 35 degrees. The thin-hit approximation of 70% still provides a good fit. Here are the equations summarizing the new approximations:

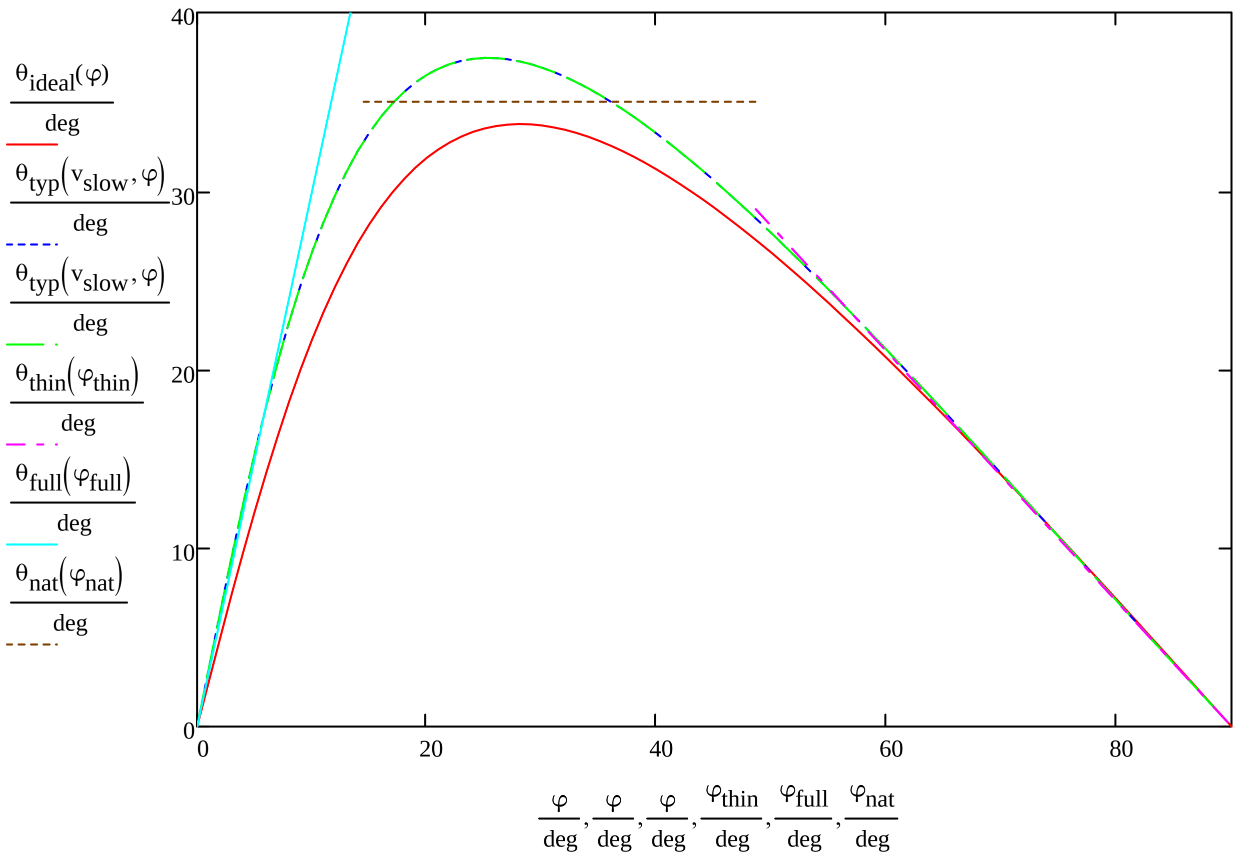
$$\theta_{full}(\varphi) := 3\cdot\varphi \qquad \theta_{nat}(\varphi) := 35\cdot deg$$

Notice how shot speed only has a slight effect on the results.

typical CB deflection angle vs. ball-hit fraction:



typical CB deflection angle vs. cut angle:



Bottom Line:

With a rolling CB shot, the CB deflection angle will be:

- about **35 degrees** for an **average cut** shot (between a 1/4-ball to a 3/4-ball hit)
- about **70% of the angle to the tangent line** for a **thin hit** (less than a 1/4-ball hit)
- about **3-times the cut angle** for a **thick hit** (more than a 3/4-ball hit)

Also, here are exact **CB deflection angle values** for important ball-hit references:

$\theta_{typ}(v_{slow}, \varphi_f(0.5)) = 36.9 \cdot deg$ 1/2-ball hit

$\theta_{typ}(v_{slow}, \varphi_f(0.25)) = 28.5 \cdot deg$ 1/4-ball hit

$\theta_{typ}(v_{slow}, \varphi_f(0.75)) = 32.8 \cdot deg$ 3/4-ball hit