technical proof technical proof

TP B.4
Stroke speed and acceleration vs. distance

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
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originally posted: 11/3/2008 last revision: 2/10/2009

stroke length to CB: $d := 12 \cdot in$ $\quad d = 0.305\,m$

cue speed at CB impact: $v_f := 15 \cdot mph$ $\quad v_f = 6.706\,\frac{m}{s}$

cue mass: $m_c := 19 \cdot oz$ $\quad m_c = 0.539\,kg$

Kinematic/dynamic relationships:

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} \qquad \text{cue acceleration}$$

$$F = m_c \cdot a = m_c \cdot v \cdot \frac{dv}{dx} \qquad \text{force on cue}$$

Classic "coast into the ball" pendulum (p) stroke (assuming a classic 1/4-wave sinusoidal speed vs. distance curve):

$$v_p(x) := v_f \cdot \sin\left(\frac{x}{d} \cdot \frac{\pi}{2}\right)$$

$$a_p = v_p(x) \cdot v_f \cdot \frac{\pi}{2 \cdot d} \cdot \cos\left(\frac{x}{d} \cdot \frac{\pi}{2}\right) \qquad a_p(x) := \frac{v_f^{\,2}\pi}{4 \cdot d} \cdot \sin\left(\frac{\pi}{d} \cdot x\right)$$

$$F_p(x) := m_c \cdot a_p(x)$$

Typical "accelerate into the ball" (a) stroke (assuming a straight-line speed vs. distance curve):

$$v_a(x) := v_f \cdot \frac{x}{d}$$

$$a_a = v_a(x) \cdot \frac{v_f}{d} \qquad a_a(x) := \left(\frac{v_f}{d}\right)^2 \cdot x$$

$$F_a(x) := m_c \cdot a_a(x) \qquad x := 0, \frac{d}{20} .. d$$

Here are alternative (time-based) formulations of stroke kinematics for coasting-pendulum vs. accelerating strokes:

Classic "coast into the ball" pendulum (p) stroke (assuming a 1/2-wave sinusoidal speed vs. time curve):

$$v_{pa}(t) = \frac{v_f}{2}\cdot\left(1 - \cos\left(\frac{\pi}{t_{fp}}\cdot t\right)\right)$$

Integrating gives:

$$x_{pa}(t) = \frac{v_f}{2}\cdot\left(t - \frac{t_{fp}}{\pi}\cdot\sin\left(\frac{\pi}{t_{fp}}\cdot t\right)\right)$$

Evaluating boundary conditions gives:

$$x_{pa}(t_{fp}) = d = \frac{v_f}{2}\cdot\left(t_{fp} - \frac{t_{fp}}{\pi}\cdot\sin\left(\frac{\pi}{t_{fp}}\cdot t_{fp}\right)\right) \qquad d = 12\cdot in$$

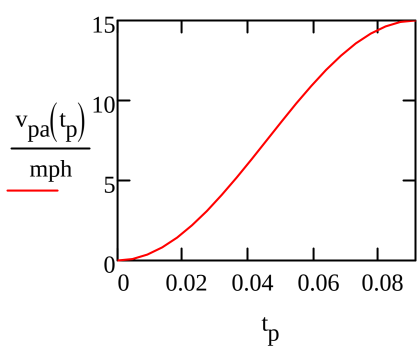
$$v_f = 15\cdot mph$$

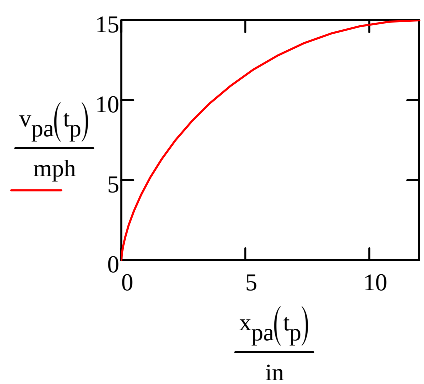
$$t_{fp} := \frac{2\cdot d}{v_f} \qquad t_{fp} = 0.091\ s$$

Therefore, the resulting velocity curve looks like:

$$v_{pa}(t) := \frac{v_f}{2}\cdot\left(1 - \cos\left(\frac{\pi}{t_{fp}}\cdot t\right)\right) \qquad x_{pa}(t) := \frac{v_f}{2}\cdot\left(t - \frac{t_{fp}}{\pi}\cdot\sin\left(\frac{\pi}{t_{fp}}\cdot t\right)\right)$$

$$t_p := 0\cdot sec, \frac{t_{fp}}{20} .. t_{fp} \qquad v_{pa}(t_{fp}) = 15\cdot mph \qquad x_{pa}(t_{fp}) = 12\cdot in$$





Differentiating the velocity equation gives us acceleration and force:

$$a_{pa}(t) := \frac{\pi\cdot v_f}{2\cdot t_{fp}}\cdot\sin\left(\frac{\pi}{t_{fp}}\cdot t\right) \qquad F_{pa}(t) := m_c\cdot a_{pa}(t)$$

Typical "accelerate into the ball" (a) stroke (assuming a 1/4-wave sinusoidal acceleration vs. time curve):

$$a_{aa}(t) = a_f \cdot \sin\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right)$$

Integrating gives:

$$v_{aa}(t) = v_f - \frac{2 \cdot a_f \cdot t_{fa}}{\pi} \cdot \cos\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right) \qquad x_{aa}(t) = v_f \cdot t - \frac{4 \cdot a_f \cdot t_{fa}^{\;2}}{\pi^2} \cdot \sin\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right)$$

Evaluating boundary conditions gives:

$$v_{aa}(0) = 0 = v_f - \frac{2 \cdot a_f \cdot t_{fa}}{\pi} \cdot \cos\left(\frac{\pi}{2 \cdot t_{fa}} \cdot 0\right) \qquad x_{aa}(t_{fa}) = d = v_f \cdot t_{fa} - \frac{2 \cdot v_f \cdot t_{fa}}{\pi} \cdot \sin\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t_{fa}\right) \qquad d = 12 \cdot in$$

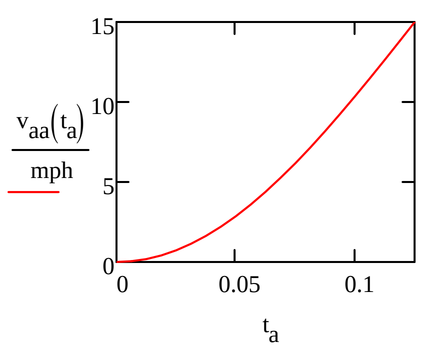
$$v_f = 15 \cdot mph$$

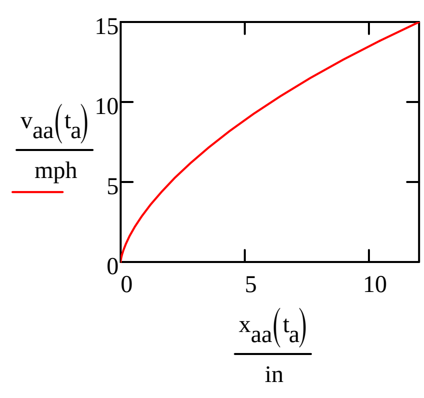
$$a_f = \frac{\pi \cdot v_f}{2 \cdot t_{fa}} \qquad t_{fa} := \frac{d}{v_f \cdot \left(1 - \frac{2}{\pi}\right)} \qquad t_{fa} = 0.125\ s$$

Therefore, the resulting velocity curve looks like:

$$v_{aa}(t) := v_f \cdot \left(1 - \cos\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right)\right) \qquad x_{aa}(t) := v_f \cdot t - \frac{2 \cdot v_f \cdot t_{fa}}{\pi} \cdot \sin\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right)$$

$$t_a := 0 \cdot sec, \frac{t_{fa}}{20} .. t_{fa} \qquad x_{aa}(t_{fa}) = 12 \cdot in \qquad v_{aa}(t_{fa}) = 15 \cdot mph$$





Differentiating the velocity equation gives us acceleration and force:

$$a_{aa}(t) := \frac{\pi \cdot v_f}{2 \cdot t_{fa}} \cdot \sin\left(\frac{\pi}{2 \cdot t_{fa}} \cdot t\right) \qquad F_{aa}(t) := m_c \cdot a_{aa}(t)$$

Here's yet **another alternative formulation** recommended by "Jal" (from the online forums), who helped me work out some of the details. This time, let's look at a sinusoidal acceleration profile resulting in the desired speed with the **smallest peak force** (pf).

The acceleration is assumed to be of the form:

$$a_{pf}(t) = \frac{F}{m_c} \cdot \sin(\omega \cdot t)$$

where F is the peak force.

Integrating twice gives speed and distance as functions of time:

$$v_{pf}(t) = \frac{F}{m_c \cdot \omega} \cdot (1 - \cos(\omega \cdot t))$$

$$x_{pf}(t) = \frac{F}{m_c \cdot \omega^2} \cdot (\omega \cdot t - \sin(\omega \cdot t))$$

From these equations, the peak force is related to the final position (d) and speed (v_f), after time t_{fpf}, according to:

$$F(\omega, t_{fpf}) = m_c \cdot d \cdot \frac{\omega^2}{(\omega \cdot t_{fpf} - \sin(\omega \cdot t_{fpf}))} = m_c \cdot v_f \cdot \frac{\omega}{1 - \cos(\omega \cdot t_{fpf})}$$

Defining $z = \omega t_{fpf}$, and solving for ω in term of z gives:

$$\omega(z) = \frac{v_f}{d} \cdot \frac{(z - \sin(z))}{(1 - \cos(z))}$$

Substituting this back into the speed peak-force equation gives:

$$F(z) = \frac{m_c \cdot v_f^2}{d} \cdot \frac{(z - \sin(z))}{(1 - \cos(z))^2}$$

To get the desired final speed (v_f) over the stroke length (d) with the minimum peak force,

$$\frac{d}{dz} F(z) = 0 = \frac{d}{dz}\left[\frac{(z - \sin(z))}{(1 - \cos(z))^2}\right]$$

giving:

$$(1 - \cos(z))^2 \cdot (1 - \cos(z)) - (z - \sin(z)) \cdot 2 \cdot (1 - \cos(z)) \sin(z) = 0$$

$$(1 - \cos(z))^2 - 2 \cdot \sin(z) \cdot (z - \sin(z)) = 0$$

The solution to this equation is:

$z := 90 \cdot deg$ Given

initial guess

$$(1 - \cos(z))^2 - 2 \cdot \sin(z) \cdot (z - \sin(z)) = 0$$

$$z := Find(z) = 113.145 \cdot deg$$

verifying: $(1 - \cos(z))^2 - 2 \cdot \sin(z) \cdot (z - \sin(z)) = 0$

so, from the equations for ω, z, and F above,

$$\omega := \frac{v_f}{d} \cdot \frac{(z - \sin(z))}{(1 - \cos(z))} = 16.665 \frac{rad}{sec}$$

$$t_{fpf} := \frac{z}{\omega} = 0.118\ s$$

$$F := \frac{m_c \cdot v_f^{\ 2}}{d} \cdot \frac{(z - \sin(z))}{(1 - \cos(z))^2} = 9.714\ lbf$$

Giving:

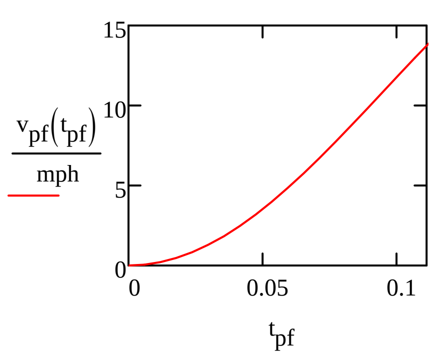
$$x_{pf}(t) := \frac{F}{m_c \cdot \omega^2} \cdot (\omega \cdot t - \sin(\omega \cdot t))$$

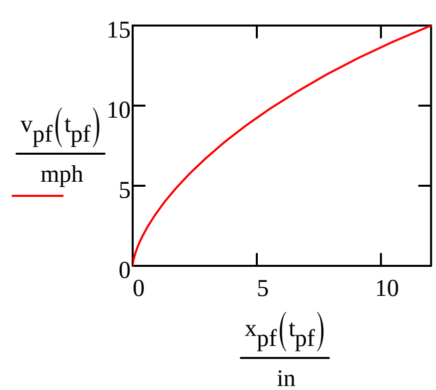
$$a_{pf}(t) := \frac{F}{m_c} \cdot \sin(\omega \cdot t)$$

$$v_{pf}(t) := \frac{F}{m_c \cdot \omega} \cdot (1 - \cos(\omega \cdot t))$$

$$F_{pf}(t) := m_c \cdot a_{pf}(t)$$

$$t_{pf} := 0 \cdot sec, \frac{t_{fpf}}{20} .. t_{fpf}$$

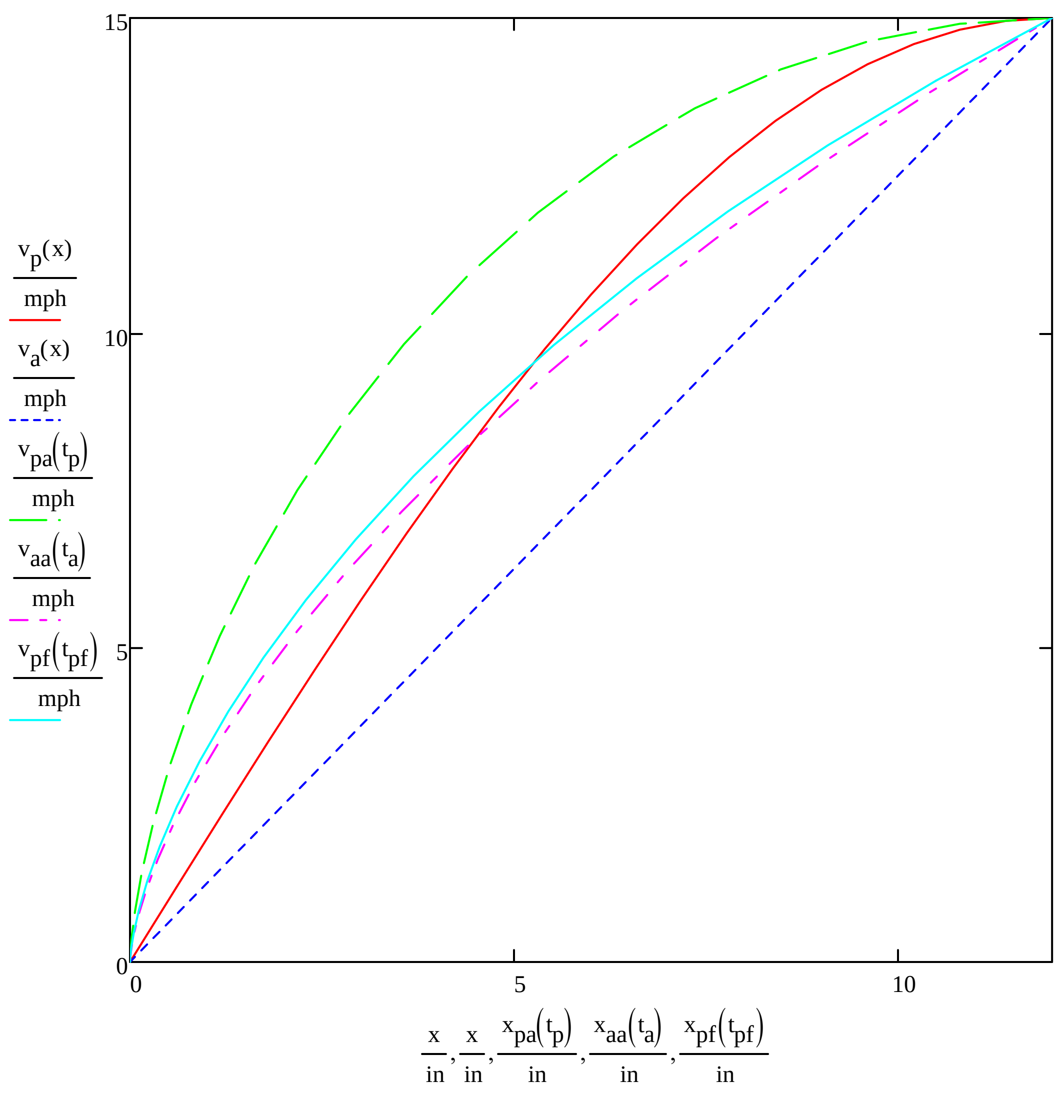




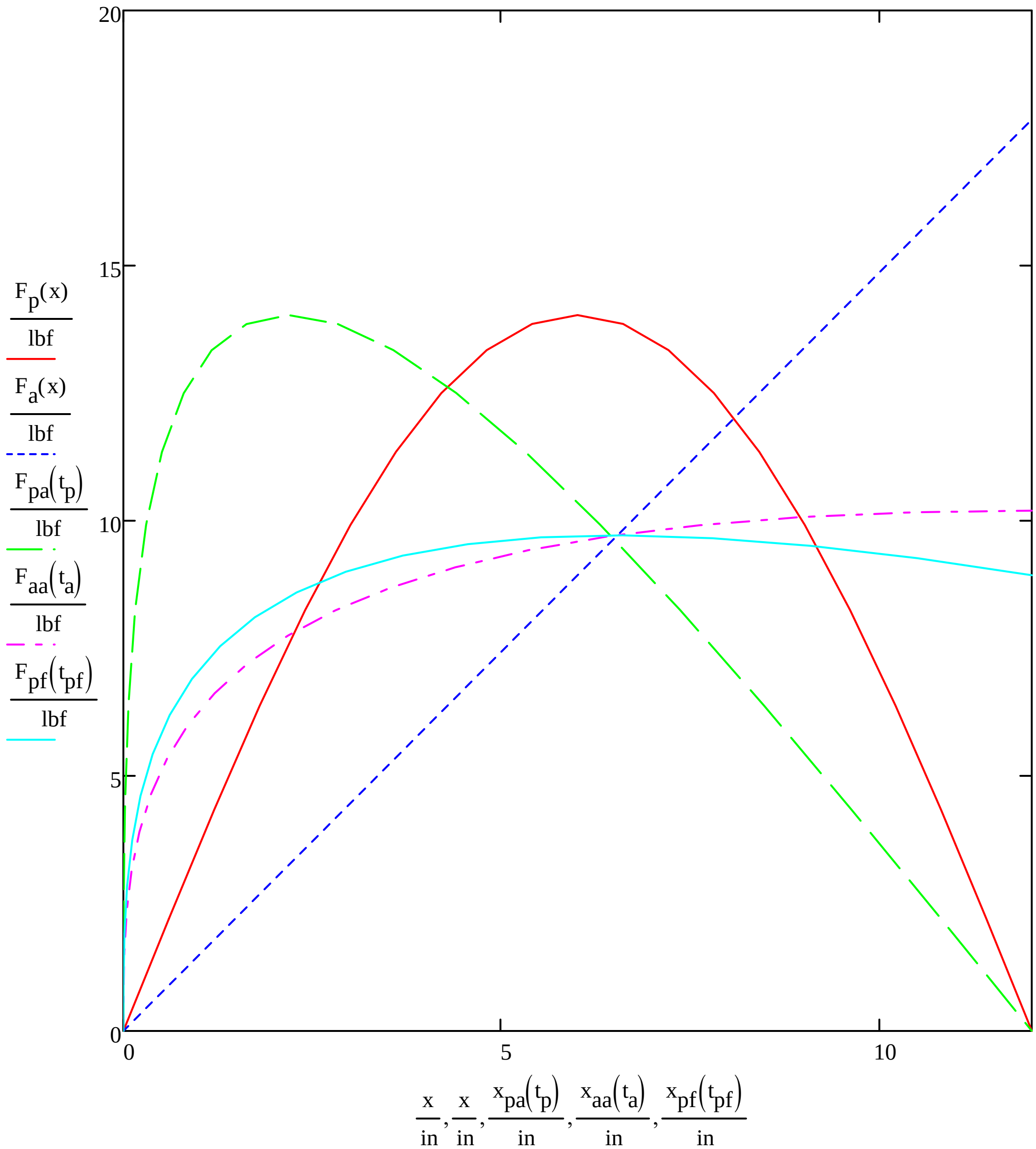
Now comparing the results from all of the models:

p: pendulum (1/4-wave sinusoidal v vs. x)
a: accelerating (straight-line v vs. x)
pa: pendulum - alternative (1/2-wave sinusoidal v vs. t)
aa: accelerating - alternative (1/4-wave sinusoidal a vs. t)
pf: accelerating - alternative (minimum peak force)

Speed vs. distance:



Force vs. distance:



speed vs. bridge length change:

$x_{min} := 0.9 \cdot d = 10.8 \cdot in$ $\quad x_{max} := 1.1 \cdot d = 13.2 \cdot in$ $\quad x := x_{min}, x_{min} + \frac{d}{20} .. x_{max}$

$t := t_{fp}$

Given

$x_{pa}(t) = x_{min}$

$t_{p\_min} := Find(t) = 0.086\, s$ $\quad x_{pa}(t_{p\_min}) = 10.8 \cdot in$

$t := t_{fp}$

Given

$x_{pa}(t) = x_{max}$

$t_{p\_max} := Find(t) = 0.095\, s$ $\quad x_{pa}(t_{p\_max}) = 13.2 \cdot in$

$$\Delta t := \frac{(t_{p\_max} - t_{p\_min})}{20} \qquad t_p := t_{p\_min}, t_{p\_min} + \Delta t .. t_{p\_max}$$

$t := t_{fa}$

Given

$x_{aa}(t) = x_{min}$

$t_{a\_min} := Find(t) = 0.12\, s$ $\quad x_{aa}(t_{a\_min}) = 10.8 \cdot in$

$t := t_{fa}$

Given

$x_{aa}(t) = x_{max}$

$t_{a\_max} := Find(t) = 0.13\, s$ $\quad x_{aa}(t_{a\_max}) = 13.2 \cdot in$

$$\Delta t := \frac{(t_{a\_max} - t_{a\_min})}{20} \qquad t_a := t_{a\_min}, t_{a\_min} + \Delta t .. t_{a\_max}$$

$t := t_{fpf}$

Given

$x_{pf}(t) = x_{min}$

$t_{pf\_min} := Find(t) = 0.114\, s$ $\quad x_{pf}(t_{pf\_min}) = 10.8 \cdot in$

$t := t_{fpf}$

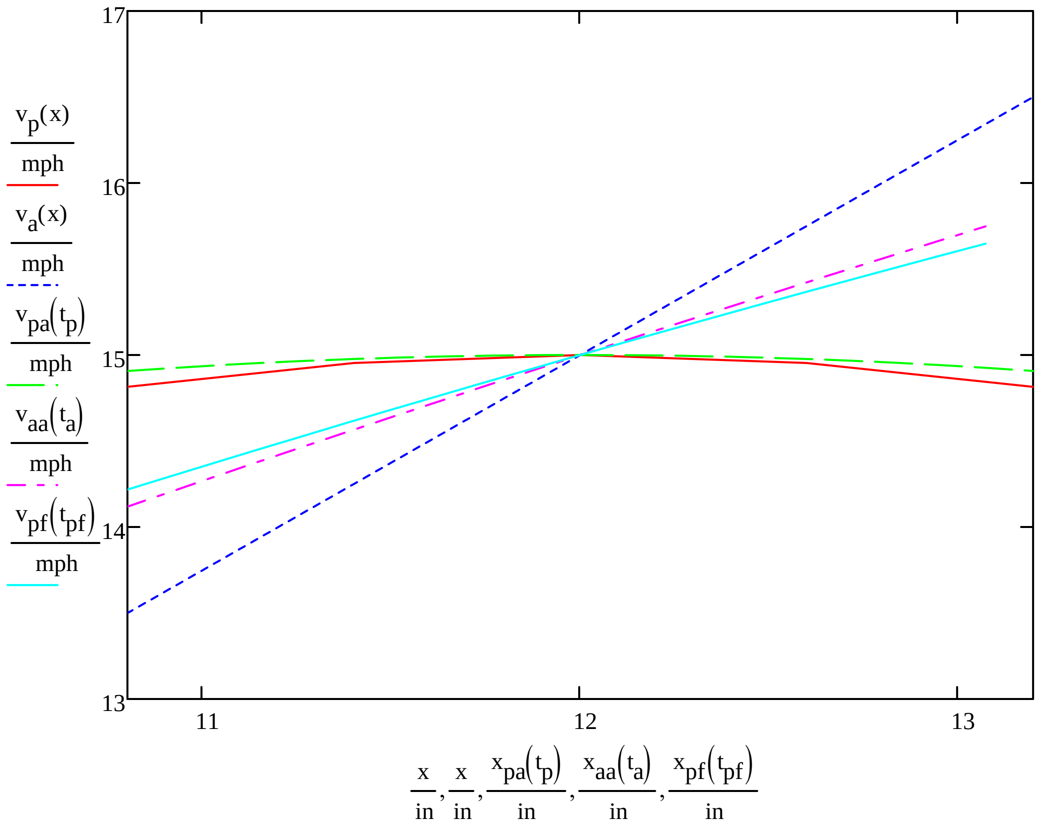
Given

$x_{pf}(t) = x_{max}$

$t_{pf\_max} := Find(t) = 0.123\, s$ $\quad x_{pf}(t_{pf\_max}) = 13.2 \cdot in$

$$\Delta t := \frac{(t_{pf\_max} - t_{pf\_min})}{20} \qquad t_{pf} := t_{pf\_min}, t_{pf\_min} + \Delta t .. t_{pf\_max}$$

speed vs. bridge or stroke length change:



With typical pendulum (p) strokes, the speed is more constant (i.e., leveled-off) at CB impact, possibly making it easier to control shot speed, because the speed is less sensitive to variations in bridge and stroke length. With typical "accelerate into the ball" (a) strokes, the force increases and levels off during the stroke, and force is being applied all of the way up to ball impact. With a classic pendulum stroke, it is natural to coast into the ball with no force at impact. The peak force is typically lower with an "accelerate into the ball" stroke than with a pendulum stroke (for the same shot speed) because force is applied over a larger distance. Therefore, for some people, this type of stroke might seem to require less effort for a given speed, and higher speeds might be possible. A typical "accelerate into the ball" stroke usually involves more of a "piston-like" stroke, with shoulder motion and elbow drop, allowing some people to generate force more easily throughout the stroke. One disadvantage of a piston stroke is that tip-contact-point accuracy might be more difficult to control.

For actual stroke examples with video demonstrations and plots, see:

HSV B.40 - stroke speed and acceleration analysis, with Bob Jewett

For plots and analysis of actual accelerometer readings, see:

TP A.9 - Cue accelerometer measurements