Introduction to the study of rolling friction

A. Doménech, T. Doménech, and J. Cebrián
IB-Buñol, ICE-University of Valencia, Valencia, Spain

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The introduction to rolling friction is studied in relation to the height that a ball attains rolling up an inclined track after rolling down a given height along another inclined track opposite from the first. The expressions obtained for a ball rolling down a rectangular inclined track and along a horizontal track are applied to the study of the distance run along two opposed inclined tracks. Different procedures are proposed to determine the coefficient of rolling friction of the ball with the track. The values calculated from the experimental data suggest their dependence with the ball radius.

I. INTRODUCTION

In introductory mechanics courses, a usual problem is the prediction of the height h that a sphere or cylinder will attain on an inclined track after rolling down—from initial height h_0—a second inclined track opposite the former.

This problem is a direct example of the energy conservation principle in the ideal case of a cylinder or sphere that rolls without rolling friction.

A simple experiment is depicted in Fig. 1 in which a ball rolls down a track of rectangular cross section along two opposite slanted planes. The heights reached by the sphere or cylinder in a second track are always lower than the initial. Experimental results clearly are in conflict with theoretical prediction in an ideal case ($h = h_0$).

This experiment can be used as a didactic method within the context of "guided discovery." A research procedure can be proposed to students following these steps: (a) elaboration of the formal model of rolling friction, (b) application to the different situations, (c) design of adequate experiments and treatment of data, (d) calculation of the coefficient of rolling friction and comparison of the values obtained by the different methods, and (e) study of the factors influencing these coefficients.

Rolling friction is not frequently included in general physics textbooks. However, a more realistic description of rolling motion requires inclusion of this subject. The example discussed above and deceleration of a sphere rolling on a horizontal track, needs to include rolling friction.

II. THEORY

The classic treatment of rolling friction is due to Reynolds[1] (1876), Hertz[2] (1886), and Heatcote[3] (1921). Recent practical applications can be found in engineering mechanics.[4,5]

Rolling friction appears as a torque opposed to spheres' or cylinder's rotation:

$$T_f = \rho N,$$

where N is the normal force exerted by the plane on the sphere and ρ is the coefficient of rolling friction. To a first approximation, ρ can be considered dependent on the nature and state of the surfaces in contact, but not on the radius or velocity of the sphere, that are also cited as influencing factors.

This frictional torque can be justified if we assume a small depression in the contacting surfaces, so that the reaction force of the plane (see Fig. 2) is applied to the sphere at a point slightly in front of the ideal geometric point of contact. Then ρ represents the arm of the pair of forces applied on the sphere perpendicular to the sustaining plane. ρ has dimensions of length and its value must be very small in relation to the sphere's radius.

Papers of Hertz and Bentall and Johnson[6] relate surface depression to elastic properties of sphere and plane materials. Theoretical treatment of rolling friction by these authors predicts, as suggested by Fig. 2, the dependence of ρ with sphere radius and effective load on the contact point. Values of ρ cited in literature are in the range[7] 10^{-2}–10^{-3} cm, however nonaccordance exist between different authors[8]; dependence with previously mentioned factors is also neglected. Development of an experimental method of measurement of this coefficient can be interesting for engineering studies.

A. Rolling down an inclined track

In recent papers, Shaw and Wunderlich[9] and Chaplin and Miller [10] study the rolling ball and also the combination of rolling and slipping when the ball rolls down and inclined track. Including rolling friction, dynamic equations that describe the movement of a rolling ball down a track inclined θ degrees with respect to the horizontal plane will be

$$ma = mg \sin \theta - F_f,$$

$$I\alpha = R_e F_f - T_f,$$

where a is the acceleration of the ball's mass center, α the angular acceleration, m is the ball's mass, and I ($= 2mR_b^2/5$) the moment of inertia with respect to the diameter. F_f is the friction force, necessary for rolling, that if applied to the instantaneous rotation axis will not produce work and therefore does not dissipate mechanical en-

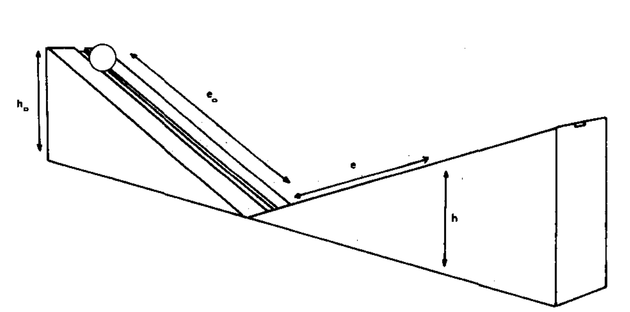

Fig. 1. The ball rolls from a height h_0 on the first plane to a height $h < h_0$ on the second, covering distances e_0 and e, respectively.

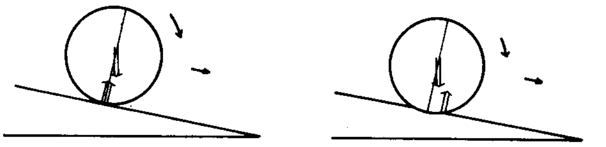

Fig. 2. Shows the pair of forces that determines the friction torque T_f; the normal N reaction of the plane on the ball and the normal component of the weight. The coefficient of the rolling friction is the arm of the pair of forces.

ergy. This requires that the deformation of the contacting surfaces must be very small.

A schematic diagram of the ball placed on a track is shown in Fig. 3; R_b is the ball's radius and R_e is the effective radius that depends on the former and on the track's width.

If we consider the case of rolling $a = \alpha R_e$ and that $N = mg(R_b/R_e)\cos\theta$, we obtain the following expression:

$$a = g\left(\frac{\sin\theta - (\rho/R_e)(R_b/R_e)\cos\theta}{1 + (2/5)(R_b/R_e)^2}\right). \qquad (1)$$

On the other hand, the ball would start rolling down the track at the limiting inclination angle θ whose tangent is given by

$$\tan\theta = (\rho^*/R_e)(R_b/R_e). \qquad (2)$$

However in this case ρ^* would represent the coefficient of static rolling friction: a distinction similar to that used with the coefficients of slipping friction.

The total torque has two components: $T_f = \rho N$ and $T_r = R_e F_f$; their ratio can easily be obtained:

$$\frac{T_f}{T_r} = \frac{1 + (2/5)(R_b/R_e)^2}{1 + (2/5)(R_b/\rho)\tan\theta}. \qquad (3)$$

When the plane inclination angle becomes larger than a given limit value (θ_L), rolling and slipping exist simultaneously; then $a \neq \alpha R_e$ and the friction force attains the limiting value[11]: $F_f = \mu mg(R_b/R_e)\cos\theta$, where μ is the dynamic coefficient of slipping friction. Acceleration of the ball's mass center is $a = g[\sin\theta - \mu(R_b/R_e)\cos\theta]$ and the relation T_f/T_r is given by

$$\frac{T_f}{T_r} = \frac{\rho N}{R_e \mu N} = \frac{\rho}{R_e \mu}, \qquad (4)$$

and the angle of transition from rolling to rolling and slipping is

$$\tan\theta_L = \frac{\mu[1 + (2/5)(R_b/R_e)^2] - \rho/R_e}{(2/5)(R_b/R_e)}. \qquad (5)$$

Conditions for ball's rolling, rolling and slipping, and only slipping are summarized in Table I.

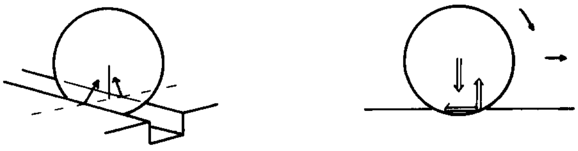

Fig. 3. The effective radius R_e of the ball corresponds to the distance between its mass center and the instantaneous rotation axis (dotted line).

Table I. Conditions for ball's rolling, rolling and slipping, and only slipping.

Repose	$\tan\theta < (\rho^*/R_e)(R_b/R_e)$ and $\mu^*(R_b/R_e)$
Rolling	$(\rho^*/R_e)(R_b/R_e) < \tan\theta < \mu^*(R_b/R_e)$
Slipping and rolling	$\tan\theta > (\rho^*/R_e)(R_b/R_e)$ and $\mu^*(R_b/R_e)$
Slipping	$\mu^*(R_b/R_e) < \tan\theta < (\rho^*/R_e)(R_b/R_e)$

(μ^*: static coefficient of slipping friction).
(ρ^*: static coefficient of rolling friction).

B. Rolling along a horizontal plane

If we consider the case of a ball that rolls along a horizontal track, a constant deceleration will exist due to rolling friction if we assume, as we have implicitly done until now, that the rail and the ball are homogeneous, that the width track is constant, that the ball is a perfect sphere and that the coefficient of friction is also constant.

We can deduce the deceleration from the following equations:

$$ma = F_f,$$

$$Ia = T_f - R_e F_f,$$

(see Fig. 3).

As $a = \alpha R_e$ and now $N = mg(R_b/R_e)$, we can propose that the acceleration of the ball's mass center is

$$a = g\left(\frac{(\rho/R_e)(R_b/R_e)}{1 + (2/5)(R_b/R_e)^2}\right), \qquad (6)$$

and relation T_f/T_r is $1 + (2/5)(R_b/R_e)^2$.

C. Rolling between two inclined planes

If we let a ball roll down a slanted plane with no slipping from a height h_0 it will run a distance e_0 on the plane

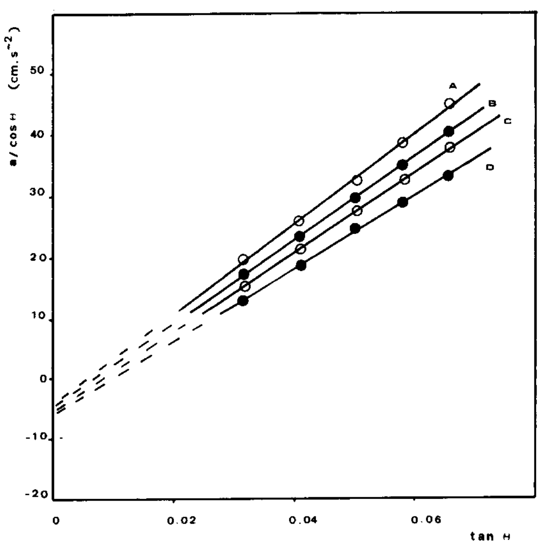

Fig. 4. Dependence of the acceleration with the inclination angle; steel balls, wood track. Diameters: A: 2.500 cm, B: 1.510 cm, C: 1.360 cm, D: 1.125 cm.

Table II. Coefficients of rolling friction calculated from inclined track (i), horizontal track (ii), and double track (iii) methods. Column (iv) shows static values calculated from minimum rolling angles.

Ball	Diameter (cm)	ρ(cm)×10^3 (i)	(ii)	(iii)	(iv)
Aluminum track					
Steel	1.125	1.4 ± 0.5	1.0 ± 0.4	2.0 ± 0.4	2.2 ± 0.4
Steel	1.270	1.8 ± 0.6	1.1 ± 0.4	2.8 ± 0.5	3.0 ± 0.5
Steel	1.360	1.9 ± 0.6	1.2 ± 0.4	3.0 ± 0.5	3.7 ± 0.6
Steel	1.510	3.1 ± 0.8	1.3 ± 0.4	3.8 ± 0.6	4.5 ± 0.6
Steel	2.000	3.6 ± 0.9	2.0 ± 0.5	4.8 ± 0.6	6.5 ± 0.8
Steel	2.500	4.8 ± 1.1	2.4 ± 0.6	5.0 ± 0.6	7.4 ± 0.9
Alum.	2.500	6.4 ± 1.5	3.2 ± 0.8	7.4 ± 0.8	9.8 ± 1.1
Brass	2.500	3.2 ± 0.8	2.2 ± 0.5	5.9 ± 0.7	8.2 ± 1.0
Wood track					
Steel	1.125	3.9 ± 1.2	3.5 ± 1.1	5.3 ± 1.4	6.1 ± 0.9
Steel	1.270	4.2 ± 1.2	3.8 ± 1.1	5.7 ± 1.5	7.0 ± 1.0
Steel	1.360	4.6 ± 1.3	4.2 ± 1.2	6.1 ± 1.5	7.3 ± 1.0
Steel	1.510	4.7 ± 1.3	5.8 ± 1.6	6.6 ± 1.6	7.7 ± 1.0
Steel	2.000	5.5 ± 1.4	6.1 ± 1.6	6.8 ± 1.6	9.7 ± 1.2
Steel	2.500	6.0 ± 1.4	7.8 ± 1.8	7.3 ± 1.7	10.6 ± 1.2
Alum.	2.500	4.2 ± 0.6	17 ± 3	6.5 ± 1.6	15.5 ± 1.8
Brass	2.500	6.5 ± 1.5	8.0 ± 1.8	7.5 ± 1.7	10.6 ± 1.2

($h_0 = e_0 \sin\theta_0$) reaching the plane's base with a velocity v given by equation:

$$v=\left[2ge_0\left(\frac{\sin\theta_0-(\rho/R_e)(R_b/R_e)\cos\theta_0}{1+(2/5)(R_b/R_e)^2}\right)\right]^{1/2}. \quad (7)$$

The frictional work exerted in the fall along the first track can be obtained as $W_{rf}=mgh_0-mv^2/2-I\omega^2/2$ (where $v=\omega R_e$), resulting in

$$W_{rf}=mge_0(\rho/R_e)(R_b/R_e)\cos\theta_0. \quad (8)$$

For slipping motion, frictional work is given by

$$W_{sf}=\mu mge_0(R_b/R_e)\cos\theta_0. \quad (9)$$

An analogous treatment explained in Sec. II A allows us to conclude that the ball will go up the second track with a constant deceleration given by

$$a=g\left(\frac{\sin\theta+(\rho/R_e)(R_b/R_e)\cos\theta}{1+(2/5)(R_b/R_e)^2}\right). \quad (9)$$

The distance traveled by the ball along this plane, which is inclined at an angle θ with respect to the horizontal plane is

$$\frac{e}{e_0}=\frac{\sin\theta_0-(\rho/R_e)(R_b/R_e)\cos\theta_0}{\sin\theta+(\rho/R_e)(R_b/R_e)\cos\theta}. \quad (10)$$

Equation (10) shows that the ratio of the distance covered on both planes is constant for a given ball and inclinations. The preceding equation can be simplified for the case in which both planes have the same inclination as

$$\frac{1+e/e_0}{1-e/e_0}=\frac{e_0+e}{e_0-e}=(R_e/\rho)(R_e/R_b)\tan\theta. \quad (11)$$

III. EXPERIMENT

We have studied the rolling of steel, brass, and aluminum balls of different sizes along an aluminum track of width 0.950 cm and a wood track of width 0.670 cm. Both tracks were 1.000 m long. The tracks are joined by a small curved piece in both cases. Measurements of time have been performed using a system similar to that described by Chaplin and Miller.

IV. RESULTS AND DISCUSSION

A. Rolling along an inclined track

The measurement of the time spent by the ball to reach different distances from an initial position allow the calculation of the acceleration a from the slope of the straight line obtained by plotting the distances versus the squared times. This linearity confirms that the mass center of the ball has a uniformly accelerated rectilinear movement.

According to Eq. (1), a plot of $a/\cos\theta$ vs $\tan\theta$ must produce a straight line of slope $g/[1+(2/5)(R_b/R_e)^2]$, and a y intercept of $g(\rho/R_e)(R_b/R_e)/[1+(2/5)(R_b/R_e)^2]$. In Fig. 4 the results obtained with

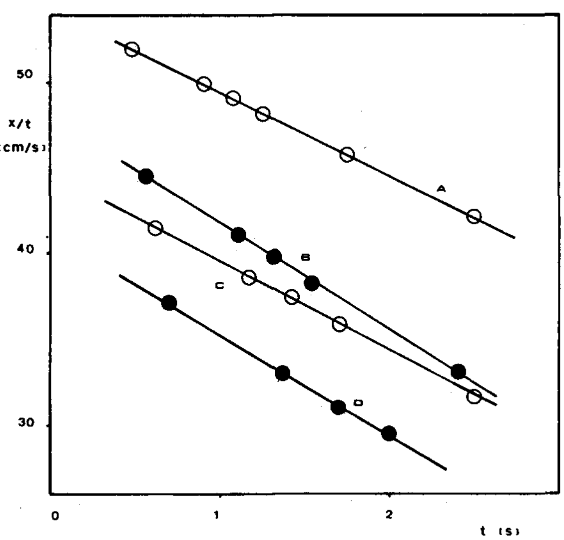

Fig. 5. Rolling along horizontal wood track; steel balls; diameters: A: 2.500 cm, B: 1.510 cm, C: 1.360 cm, D: 1.125 cm.

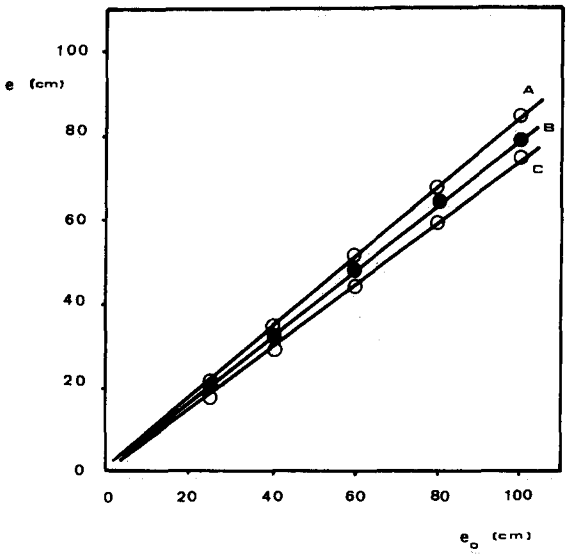

Fig. 6. Distances traveled by different steel balls on opposed inclined planes; aluminum tracks. Same inclination of both planes; $\tan\theta = 0.070$. Diameters: A: 1.270 cm, B: 1.510 cm, C: 2.000 cm.

steel balls of different diameters in wood track are plotted. The values of the coefficient of rolling friction obtained are show in Table II.

B. Rolling along a horizontal track

In this case different balls are released down a slightly inclined track and the time intervals spent to reach different distances x along a second horizontal track aligned with the former are calculated. Theoretically the movement will be rectilinear uniformly decelerated and plotting x/t vs t must correspond to straight lines according to Eq. (6).

The results for some steel balls and wood track are shown in Fig. 5. Calculated values of ρ are show in Table II.

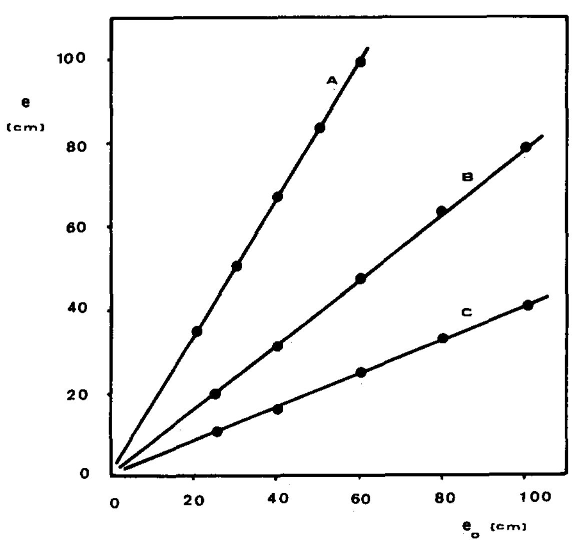

Fig. 7. Distances covered by a steel ball of diameter 2.000 cm with different inclination for each of the aluminum tracks. A: $\tan\theta_0 = 0.104$, $\tan\theta = 0.052$; B: $\tan\theta_0 = \tan\theta = 0.052$; C: $\tan\theta_0 = 0.052$, $\tan\theta = 0.104$.

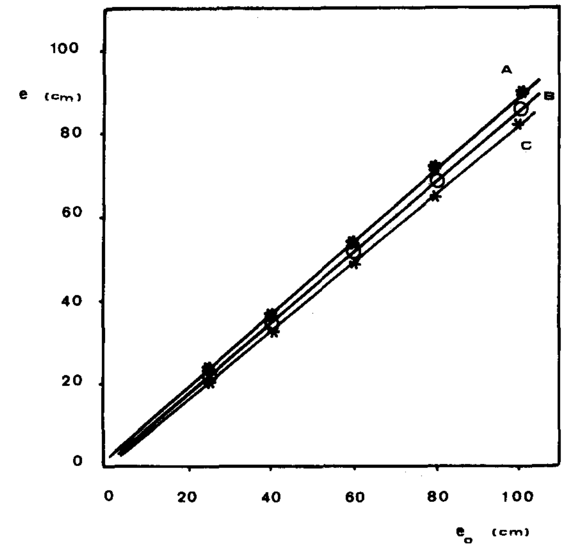

Fig. 8. Distances covered by steel (A), brass (B), and aluminum (C) balls of radius 2.500 cm. Same inclination of both aluminum planes; $\tan\theta = 0.070$.

C. Rolling along a track between two inclined planes

Equation (11) predicts that when a ball is released from a certain height using the device show in Fig. 1, the distance e_0 and e traveled by the ball along the first and second inclined plane, respectively, must keep a linear relation.

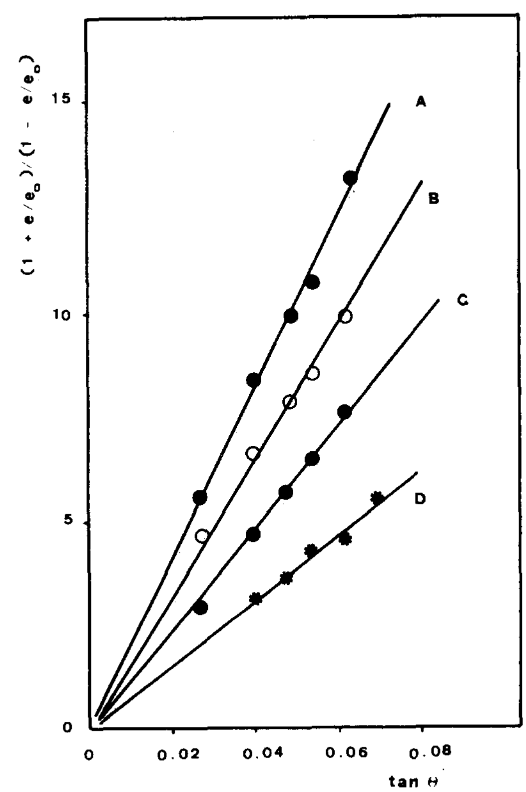

Fig. 9. Calculation of the coefficient of rolling friction by the double inclined track method. Data for the different steel ball allow the calculation of $(1 + e/e_0)/(1 - e/e_0)$ from the slope of the obtained lines e vs e_0. Aluminum tracks; diameters: A: 2.500 cm, B: 2.000 cm, C: 1.510 cm, D: 1.125 cm.

Figure 6 shows the results obtained using steel balls of different radii and Fig. 7 shows the results of rolling a ball with different track inclination. The inclination angles that have been used were always smaller than 10° in all cases.

A plot of the data obtained for steel, brass, and aluminum balls of the same radii moving between aluminum tracks with the same inclination is show in Fig. 8. The results show a different behavior of these materials attributable to their different coefficient of rolling friction.

Equation (12) has been used to calculate the values of the coefficient of rolling friction, using the data from e/e_0 ratios obtained for each ball from the slope of the lines obtained from graphic representation e vs e_0, for several inclination angles. The results are show in Fig. 9. The estimated values of the coefficient of rolling friction are comparable with those calculated in Secs. IV A and IV B. The values show an increasing variation with the ball's radius (see Table II).

V. FINAL CONSIDERATIONS

Using the values of the rolling friction coefficient shown in Table II and $\mu = 0.13$ for steel–aluminum slipping friction coefficient obtained from inclined track measurements,[9,10] Eqs. (3) and (4) suggest a small contribution of torque friction T_f to the total torque in pure rolling motion and in rolling with slipping motion along an inclined track.

According to Eq. (3), for rolling motion, this contribution decreases when the inclination angle θ increases; only for very small angles, torque friction T_f is found to have a significant contribution to the total torque. In the case $\theta = 0°$, horizontal track, T_f has an important contribution to total torque.

According to Eq. (8), the ratio of the work done by rolling friction to the initial energy would increase when angle θ decreases; large values for frictional dissipation energy can be attained for small angles.

$$W_{rf}/E_{\text{initial}} = (\rho/R_e)(R_b/R_e)\cot\theta.$$

Thus, the inclusion of rolling friction is particularly relevant in rolling motion in a horizontal track and in an inclined track for small angles of inclination.

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Ancient heliocentrists, Ptolemy, and the equant

Dennis Rawlins

Physics Department, Loyola College, Baltimore, Maryland 21210-2699

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Evidence is presented suggesting an ancient heliocentrist origin for geocentrist C. Ptolemy's planetary orbit elements and the equant. Pliny's data for Venus are shown to be inconsistent with geocentricity, and a heliocentric period-relation is found to be the basis of Ptolemy's previously unexplained and astonishingly accurate tables of the mean motion of Mars, the very planet whose orbit produced the equant. The admirable correctness of his adopted Mars elements is patently inconsistent with the ordmag 1° inaccuracy of Ptolemy's geocentric model and of his alleged empirical production.

I. THE EQUANT ADVANTAGE

In a recent paper in this Journal, J. Evans has presented a persuasive diagrammatic demonstration[1] that inevitably severe problems with Mars' orbit drove the ancients to adopt the ingenious device known as the "equant," to replace the primitive "eccentric" model. In this section, I will add some simple and useful quantitative details to Evans' argument.

Figure 1 illustrates the Ptolemaic equant model. The planet P travels uniformly (about point A) on a noneccentric circular "epicycle" (of radius r) where epicycle center A travels nonuniformly on a circular "deferent" of unit radius. The Earth T is not at the deferent's center C (thus from the geocentric perspective, the motion of A is eccentric) but is offset from it a distance e, the "eccentricity." The equant point Q (not deferent center C, as in the old eccentric model) is the center of A's uniform angular motion, where Q is placed a distance e from C, on the opposite side of C from T. (Thus, C is halfway between points T and Q.)

Before Evans' analysis, modern scholars often tacitly as-