TP A.14
The effects of cut angle, speed, and spin on object ball throw

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu

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See TPs A.5, A.6 and A.8 for background information and illustrations.

For a cue ball with both vertical-plane (draw or follow) and horizontal-plane (side) English, the velocity of the point of contact (B) between the cue ball and the object ball, at impact is:

$$\vec{v}_B = \vec{v} + \vec{\omega} \times \vec{r}_{B/O} = v\hat{j} + (\omega_x \hat{i} + \omega_z \hat{k}) \times R(-\sin(\phi)\hat{i} + \cos(\phi)\hat{j}) \tag{1}$$

So

$$\vec{v}_B = v_{B_x}\hat{i} + v_{B_y}\hat{j} + v_{B_z}\hat{k} = (-R\omega_z\cos(\phi))\hat{i} + (v - R\omega_z\sin(\phi))\hat{j} + (R\omega_x\cos(\phi))\hat{k} \tag{2}$$

Expressing this vector in tangential and normal components (see TP A.5) gives:

$$\vec{v}_B = v_{B_t}\hat{t} + v_{B_n}\hat{n} + v_{B_z}\hat{k} = (v\sin(\phi) - R\omega_z)\hat{t} + (v\cos(\phi))\hat{n} + (R\omega_x\cos(\phi))\hat{k} \tag{3}$$

Therefore, the relative "sliding" velocity vector for the point of contact between the cue ball and object ball can be expressed as:

$$\vec{v}_{rel} = v_{B_t}\hat{t} + v_{B_z}\hat{k} \tag{4}$$

because the normal (n) component, which creates the impact forces, does not contribute to the relative sliding.

The friction force on the object ball acts in the direction of the sliding velocity vector:

$$\hat{e}_\mu = \frac{\vec{v}_{rel}}{|\vec{v}_{rel}|} = \frac{v_{B_t}\hat{t} + v_{B_z}\hat{k}}{\sqrt{v_{B_t}^{\,2} + v_{B_z}^{\,2}}} = \left(\frac{v_{B_t}}{\sqrt{v_{B_t}^{\,2} + v_{B_z}^{\,2}}}\right)\hat{t} + \left(\frac{v_{B_z}}{\sqrt{v_{B_t}^{\,2} + v_{B_z}^{\,2}}}\right)\hat{k} = e_{\mu_t}\hat{t} + e_{\mu_z}\hat{k} \tag{5}$$

From Equation 8 in TP A.5, the normal impulse between the cue ball and object ball, assuming a perfectly elastic collision, is:

$$\tilde{F}_n = mv\cos(\phi) \tag{6}$$

From TP A.6, and using Equations 3 and 5, the maximum possible friction impulse component that contributes to object ball throw, based on the limitation of friction, is:

$$\tilde{F}_{t_{\max}} = \mu \tilde{F}_n e_{\mu_t} = \frac{\mu m v\cos(\phi)(v\sin(\phi) - R\omega_z)}{\sqrt{(v\sin(\phi) - R\omega_z)^2 + (R\omega_x\cos(\phi))^2}} \tag{7}$$

Another limit on the maximum possible friction impulse is based on the requirement that the relative sliding motion between the cue ball (CB) and object ball (OB) cannot reverse direction during impact. The relative tangential speed between the balls after impact is given by:

$$v_{rel} = (v_{CB_t} - \omega_{CB}R) - (v_{OB_t} + \omega_{OB}R) \tag{8}$$

where the v terms on the right side indicate post-impact speed components in the tangential direction and the ω terms indicate post-impact rotational speeds.

In the limiting case, v_{rel} will go to zero during impact (i.e., sliding ceases during impact). In this case, the maximum post-impact tangential speed the object ball can have is:

$$v_{OB_t\max} = v_{CB_t} - (\omega_{CB} + \omega_{OB})R \tag{9}$$

From TP A.5 and impulse-momentum principles, the terms in Equation 9 can be expressed as:

$$v_{OB_t\max} = \frac{\tilde{F}_{t_{\max}}}{m} \tag{10}$$

$$v_{CB_t} = v\sin(\phi) - \frac{\tilde{F}_{t_{\max}}}{m} \tag{11}$$

$$\omega_{CB} = \omega_z + \frac{R\tilde{F}_{t_{\max}}}{\left(\frac{2}{5}mR^2\right)} = \omega_z + \frac{5\tilde{F}_{t_{\max}}}{2Rm} \tag{12}$$

$$\omega_{OB} = \frac{5\tilde{F}_{t_{\max}}}{2Rm} \tag{13}$$

Substituting Equations 10 through 13 into Equation 9 yields the maximum possible friction impulse component that contributes to object ball throw, based on the relative speed kinematics constraint:

$$\tilde{F}_{t_{\max}} = \frac{m}{7}(v\sin(\phi) - R\omega_z) \tag{14}$$

Using Equations 7 and 14, we can now determine the post-impact object ball tangential speed based on which effect is the most limiting (friction or kinematics):

$$v_{OB_t} = \frac{\tilde{F}_t}{m} = \min\left\{\begin{matrix}\left(\dfrac{\mu v\cos(\phi)}{\sqrt{\left(v\sin(\phi)-R\omega_z\right)^2+\left(R\omega_x\cos(\phi)\right)^2}}\right) \\ \dfrac{1}{7}\end{matrix}\right\}\left(v\sin(\phi)-R\omega_z\right) \tag{15}$$

The normal component of the post-impact object ball velocity, using Equation 6, is:

$$v_{OB_n} = \frac{\tilde{F}_n}{m} = v\cos(\phi) \tag{16}$$

Now the object ball throw angle can be calculated with:

$$\theta_{OB} = \tan^{-1}\left(\frac{v_{OB_t}}{v_{OB_n}}\right) \tag{17}$$

Typical values for the parameters used in the equations:

$\mu := 0.06$	average coefficient of friction between the balls	
$R := \dfrac{1.125\cdot\text{in}}{\text{m}}$	ball radius converted to meters	$R = 0.029$
$v := \dfrac{3\cdot\text{mph}}{\frac{\text{m}}{\text{s}}}$	average cue ball speed converted to meters/sec	$v = 1.341$
$\omega_{roll} := \dfrac{v}{R}$	natural-roll spin rate	

Typical speeds (converted to meters/sec):

$v_{slow} := \dfrac{1\cdot\text{mph}}{\frac{\text{m}}{\text{s}}}$ $\quad v_{slow} = 0.447$

$v_{medium} := \dfrac{3\cdot\text{mph}}{\frac{\text{m}}{\text{s}}}$ $\quad v_{medium} = 1.341$

$v_{fast} := \dfrac{7\cdot\text{mph}}{\frac{\text{m}}{\text{s}}}$ $\quad v_{fast} = 3.129$

Model of friction based on Marlow data (Table 10 on p. 245 in "The Physics of Pocket Billiards," 1995)

$$vd_1 := .1\cdot\sin(45\cdot deg) \qquad \mu d_1 := 0.11$$

$$vd_2 := 1\cdot\sin(45\cdot deg) \qquad \mu d_2 := 0.06$$

$$vd_3 := 10\cdot\sin(45\cdot deg) \qquad \mu d_3 := 0.01$$

Note - the Marlow speed data is multiplied by sin(45 deg) to reflect the relative tangential speed component for the shot in his experiment (which had a 45 deg cut angle).

The friction vs. speed relation seems to follow (or let's assume it follows) a relation of the form:

$$u(v) = a + b\cdot e^{-c\cdot v}$$

We can solve for the coefficients from the set of data above using:

initial guesses:

$$a := \mu d_3 \qquad b := \mu d_1 - \mu d_3 \qquad c := \frac{-1}{vd_2}\cdot\ln\left(\frac{\mu d_2 - a}{b}\right)$$

$$a = 0.01 \qquad b = 0.1 \qquad c = 0.98$$

Given

$$\mu d_1 = a + b\cdot e^{-c\cdot vd_1}$$

$$\mu d_2 = a + b\cdot e^{-c\cdot vd_2}$$

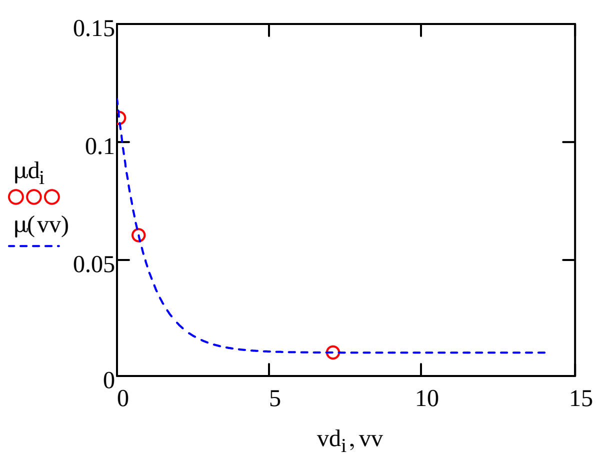
$$\mu d_3 = a + b\cdot e^{-c\cdot vd_3}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} := \mathrm{Find}(a, b, c) \qquad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9.951\times 10^{-3} \\ 0.108 \\ 1.088 \end{pmatrix}$$

$$\mu(v) := a + b\cdot e^{-c\cdot v}$$

Plot of experimental friction data with theoretical curve fit

$$i := 1..3 \qquad vv := 0, 0.01..2\cdot vd_3$$



MathCAD formulations of Equations 15 through 17, using the above formulation for friction:

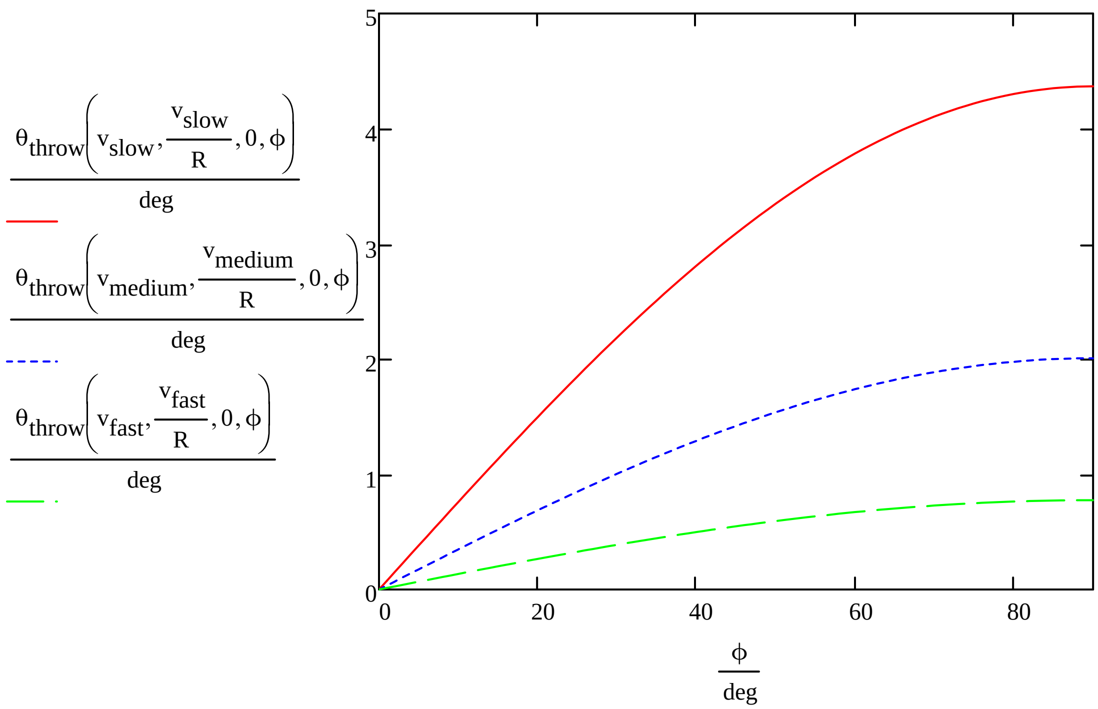
$$v_{rel}(v, \omega_x, \omega_z, \phi) := \sqrt{(v \cdot \sin(\phi) - R \cdot \omega_z)^2 + (R \cdot \omega_x \cdot \cos(\phi))^2}$$

$$\theta_{throw}(v, \omega_x, \omega_z, \phi) := \mathrm{atan}\left[\frac{\mathrm{if}\left[(v_{rel}(v, \omega_x, \omega_z, \phi) = 0), 0, \min\left(\frac{\mu(v_{rel}(v, \omega_x, \omega_z, \phi)) \cdot v \cdot \cos(\phi)}{v_{rel}(v, \omega_x, \omega_z, \phi)}, \frac{1}{7}\right) \cdot (v \cdot \sin(\phi) - R \cdot \omega_z)\right]}{v \cdot \cos(\phi)}\right]$$

These two equations describe all throw effects (see the plots below)!

collision-induced throw vs. cut angle for various-speed natural-roll shots:

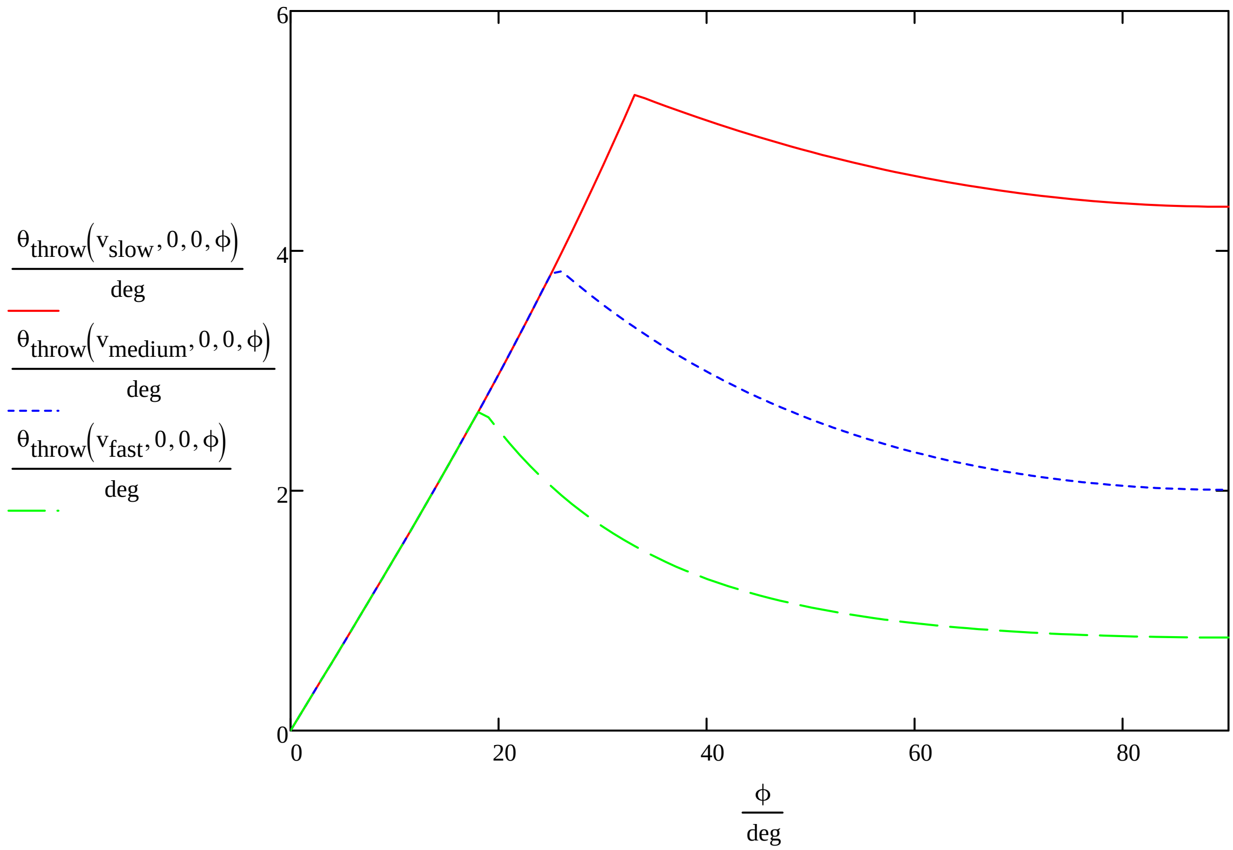
$$\phi := 0 \cdot deg, 1 \cdot deg .. 90 \cdot deg$$



<u>BOTTOM LINE</u>: slower speed results in more collision-induced throw; and the amount of throw increases with cut angle, but it levels off at higher cut angles.

collision-induced throw vs. cut angle for stun shots at various speeds:

$$\phi := 0\cdot deg, 1\cdot deg\,..\,90\cdot deg$$



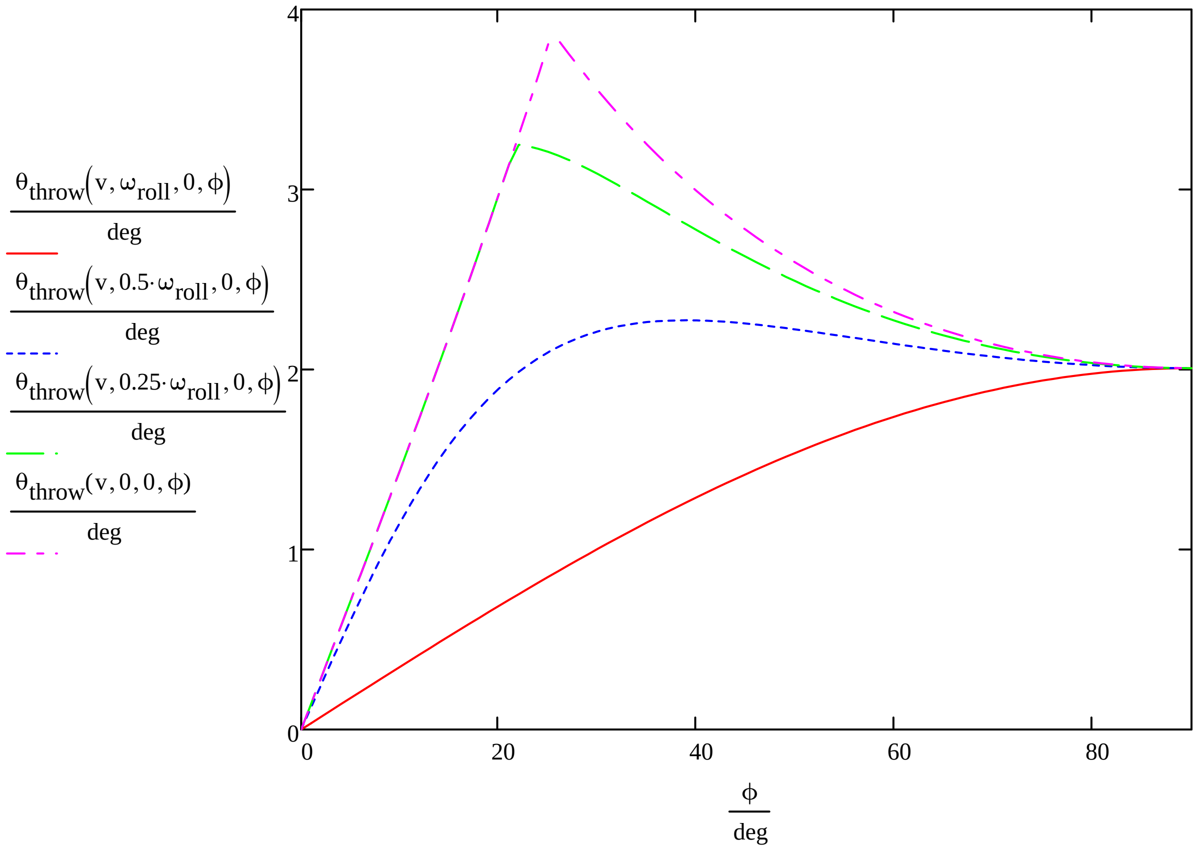
These plots can be compared to Bob Jewett's experimental data located at:

http://www.sfbilliards.com/Misc/throw.gif

and with the data presented in my September '06 instructional article. The theoretical plots and experiments agree very closely in both curve shape and throw values.

BOTTOM LINE: The amount of throw is independent of speed at small cut angles. Throw is largest in the half-ball hit range (30-degree cut angle range). At larger cut angles, throw is larger for slower speeds.

collision-induced throw vs. cut angle for shots with various amounts of vertical plane spin (follow or draw), compared to stun

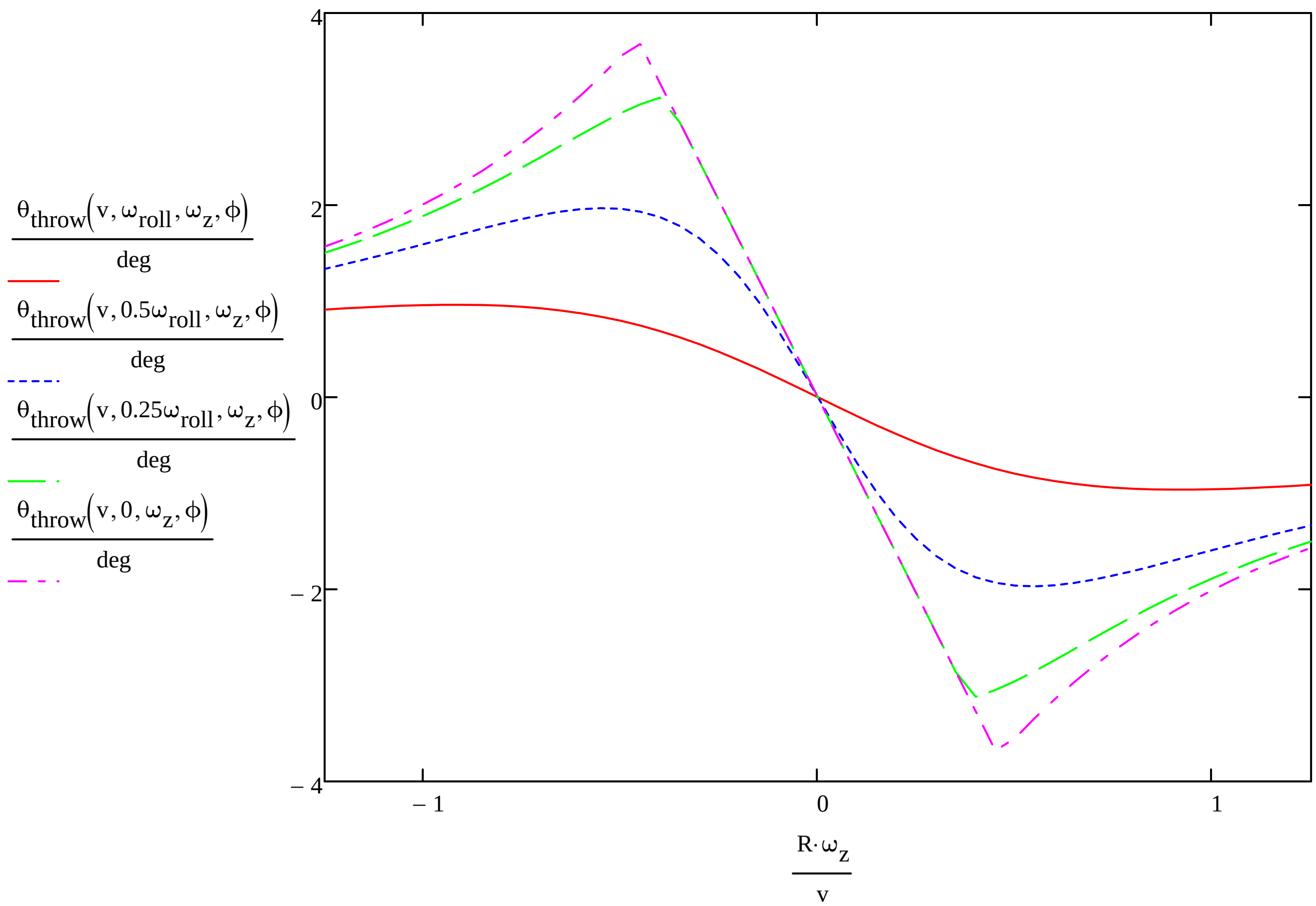


<u>BOTTOM LINE</u>: collision-induced throw is greater for stun shots close to a 1/2-ball hit (30 degree cut angle).

spin-induced throw for straight-on shots (0-degree cut angle) with various amounts of forward roll (or draw) and sidespin

$\phi := 0$ $\qquad\qquad$ $\omega_z := -1.25\omega_{roll}, -1.2\cdot\omega_{roll} .. 1.25\omega_{roll}$

(see TP A.12)



These plots are similar to those in Figure 4.4 on p. 42 of Ron Shepard's "Amateur Physics for the Amateur Pool Player," 3rd edition, 1997. However, here, a more accurate model of collision dynamics and kinematics involving friction is used.

<u>BOTTOM LINE</u>: spin-induced throw is greatest, and most sensitive to sidespin, with stun shots and for medium amounts of spin.

spin-induced throw for straight-on stun shots (0-degree cut angle) at various speeds, with different English percentages

$\phi := 0$ $\qquad\qquad$ $pE := -100\cdot\%, -95\cdot\% .. 100\%$

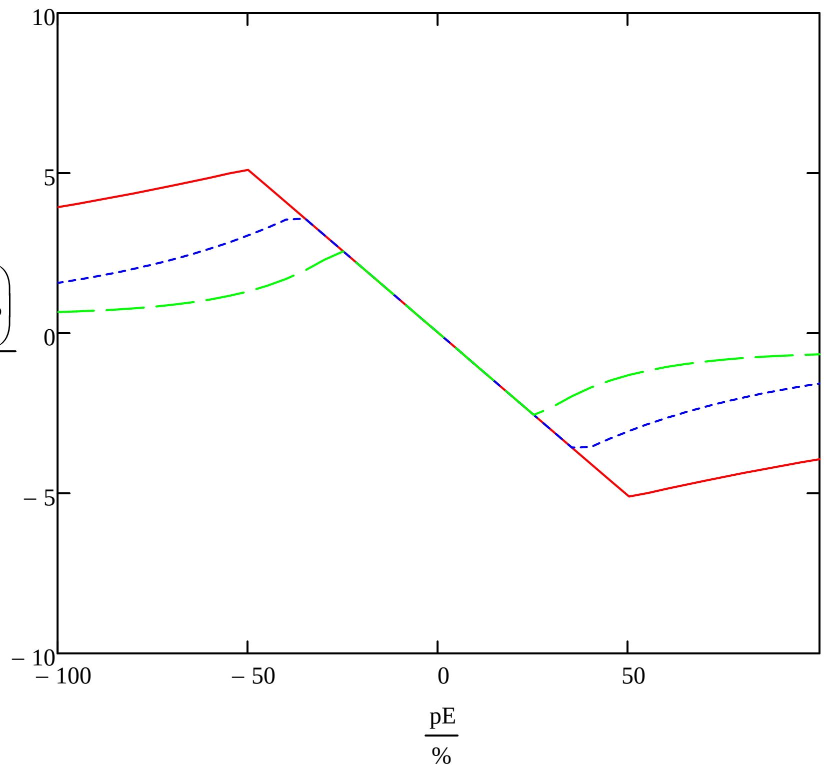
(see TP A.12 and TP A.25)

side spin = SRF (v/R) $\qquad\qquad$ SRF = 1.25 (pE)

$$\frac{\theta_{throw}\left(v_{slow}, 0, 1.25pE\cdot\frac{v_{slow}}{R}, \phi\right)}{deg}$$

$$\frac{\theta_{throw}\left(v_{medium}, 0, 1.25pE\cdot\frac{v_{medium}}{R}, \phi\right)}{deg}$$

$$\frac{\theta_{throw}\left(v_{fast}, 0, 1.25pE\cdot\frac{v_{fast}}{R}, \phi\right)}{deg}$$

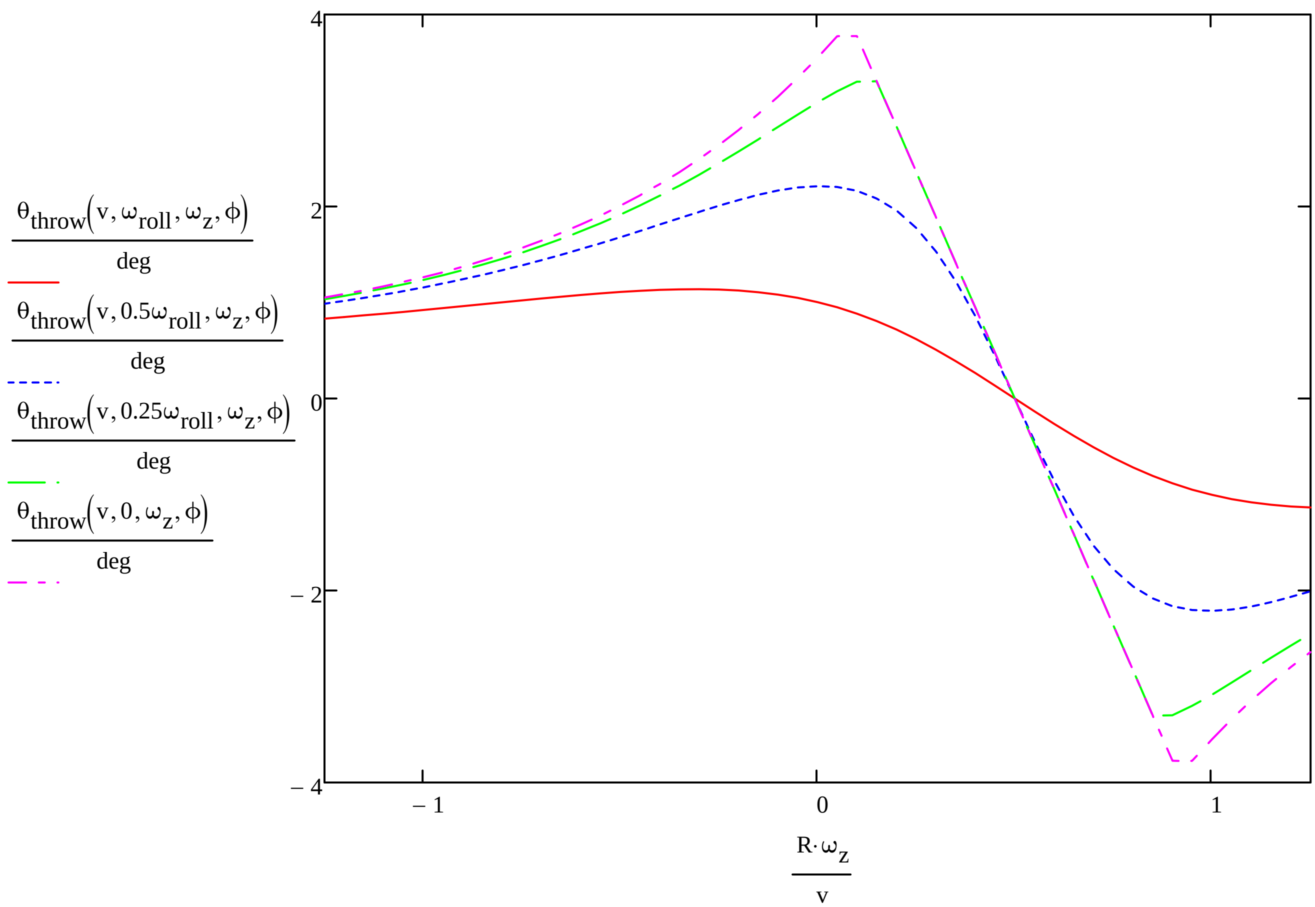


<u>BOTTOM LINE</u>: spin-induced throw is independent of speed for small amounts of English and largest at about 50% English for a slow shot. After a point, more English doesn't create more throw.

combination of collision- and spin-induced throw for a half-ball hit (30-degree cut angle) with various amounts of forward roll (or draw) and sidespin

$\phi := 30 \cdot deg$ $\qquad$ $\omega_z := -1.25\omega_{roll}, -1.2\cdot\omega_{roll} .. 1.25\omega_{roll}$

(see TP A.12)

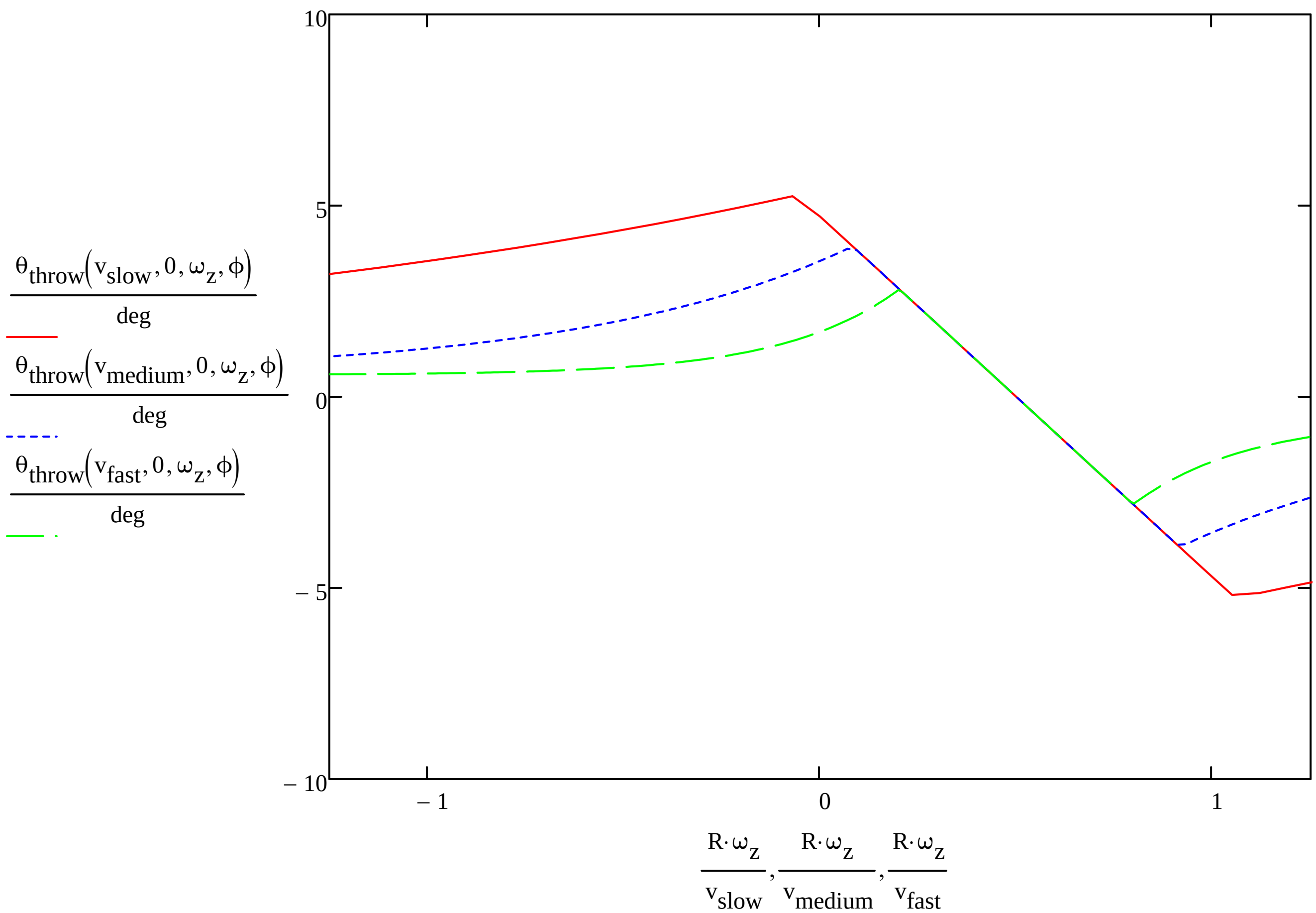


<u>BOTTOM LINE</u>: For a half-ball hit, throw is greatest for a stun shot with no sidespin or with outside English with a spin rate factor of 1 (see TP A.12). For outside English with a spin rate factor of 0.5, there is no throw.

combination of collision- and spin-induced throw for a half-ball hit (30-degree cut angle) stun shots at different speeds and with various amounts of sidespin

$\phi := 30 \cdot deg \qquad \omega_{roll} := \frac{v_{fast}}{R} \qquad \omega_z := -1.25\omega_{roll}, -1.24 \cdot \omega_{roll} .. 1.25\omega_{roll}$

(see TP A.12)



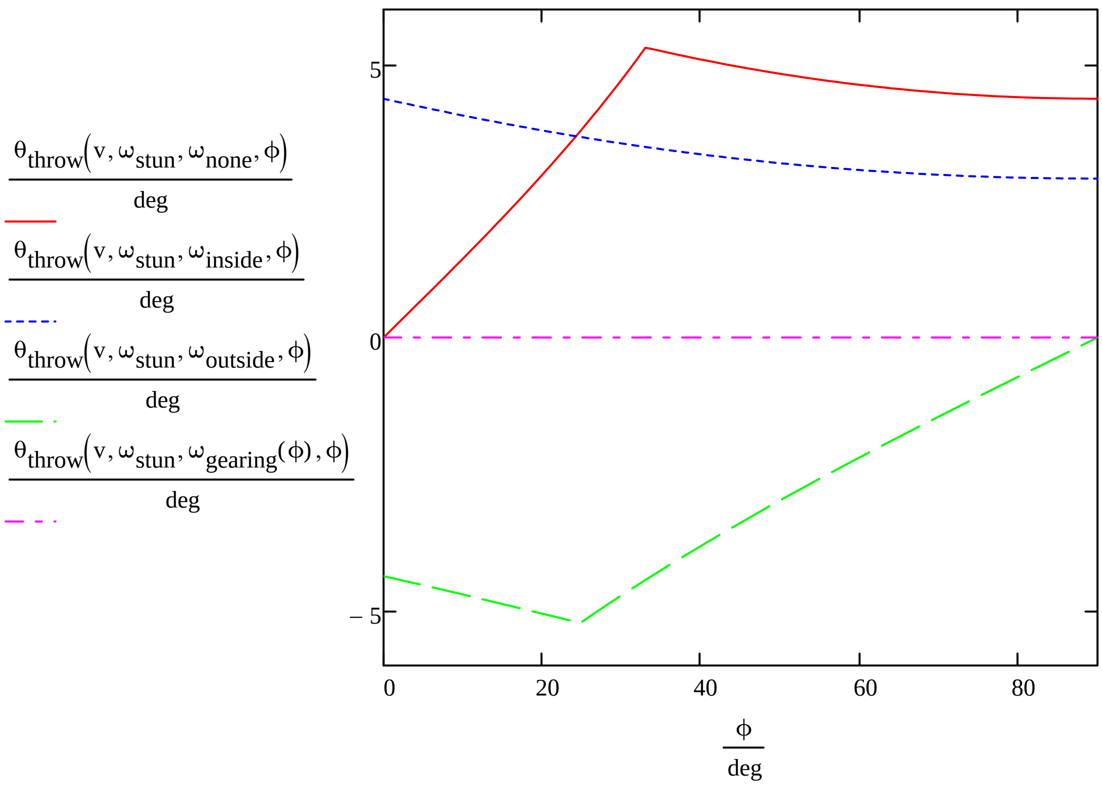
BOTTOM LINE: For a half-ball hit stun shot, throw is greatest with little or no sidespin or with significant outside English. The amount of throw is most sensitive to the amount of sidespin near the throw-cancelling outside English point.

throw vs. cut angle for various types of typical side-English, slow speed, stun shots:

$$\phi := 0 \cdot deg, 1 \cdot deg .. 90 \cdot deg$$

$$v := v_{slow} \qquad \omega_{roll} := \frac{v}{R} \qquad \omega_{stun} := 0$$

$$\omega_{none} := 0 \qquad \omega_{inside} := -\omega_{roll} \qquad \omega_{outside} := \omega_{roll} \qquad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$$



These plots agree with the qualitative plots in Figure 6-15 on p. 74 of Jack Koehler's "The Science of Pocket Billiards," 1989.

<u>BOTTOM LINE</u>: Inside English increases throw at small cut angles. Outside English reverses collision-induced throw, and has maximum effect at small cut angles. "Gearing" outside English results in absolutely no throw.

throw vs. cut angle for various types of 25% side-English, slow speed, stun shots:

$$v := v_{slow} \qquad \omega_{roll} := \frac{v}{R} \qquad \omega_{stun} := 0 \qquad SRF := \frac{5}{4} \cdot 25\%$$

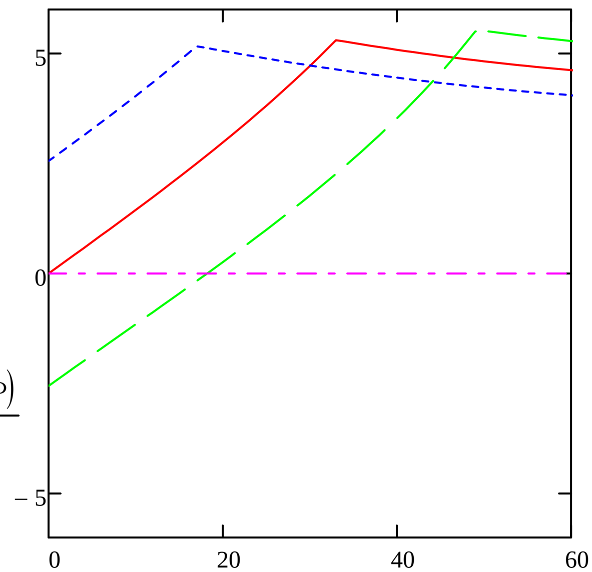
$$\omega_{none} := 0 \qquad \omega_{inside} := -SRF\,\omega_{roll} \qquad \omega_{outside} := SRF\,\omega_{roll} \qquad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$$

$\dfrac{\theta_{throw}(v, \omega_{stun}, \omega_{none}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{stun}, \omega_{inside}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{stun}, \omega_{outside}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{roll}, \omega_{gearing}(\phi), \phi)}{deg}$



$\dfrac{\phi}{deg}$

throw vs. cut angle for various types of 25% side-English, medium speed, follow/draw shots:

$$v := v_{fast} \qquad \omega_{roll} := \frac{v}{R} \qquad \omega_{stun} := 0 \qquad SRF := \frac{5}{4} \cdot 25\%$$

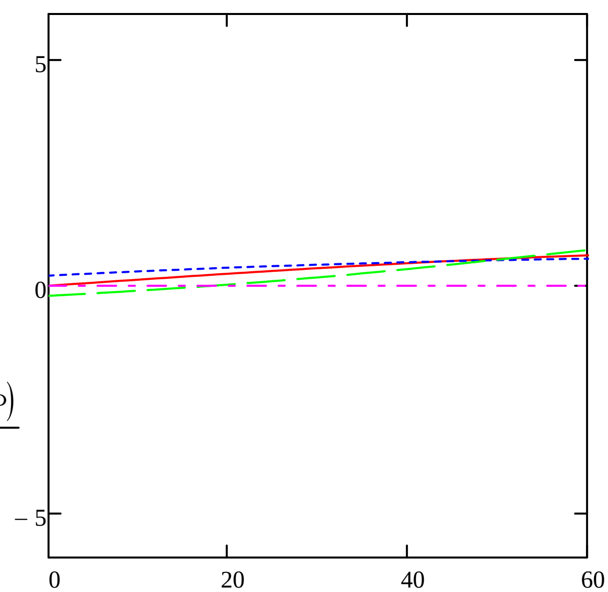
$$\omega_{none} := 0 \qquad \omega_{inside} := -SRF\,\omega_{roll} \qquad \omega_{outside} := SRF\,\omega_{roll} \qquad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$$

$\dfrac{\theta_{throw}(v, \omega_{roll}, \omega_{none}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{roll}, \omega_{inside}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{roll}, \omega_{outside}, \phi)}{deg}$

$\dfrac{\theta_{throw}(v, \omega_{roll}, \omega_{gearing}(\phi), \phi)}{deg}$

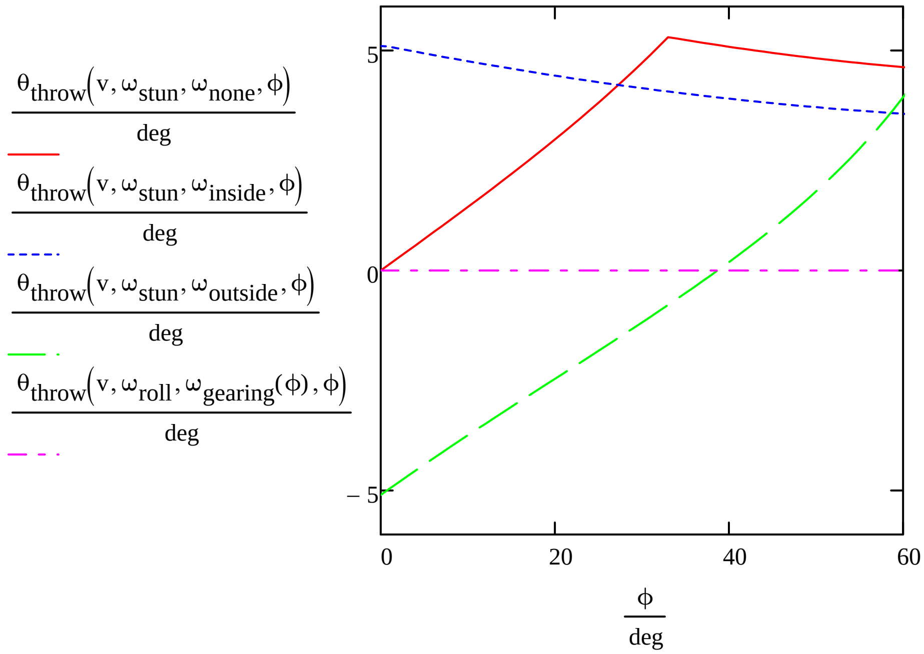


$\dfrac{\phi}{deg}$

throw vs. cut angle for various types of 50% side-English, slow speed, stun shots:

$v := v_{slow}$ $\qquad \omega_{roll} := \frac{v}{R}$ $\qquad \omega_{stun} := 0$ $\qquad SRF := \frac{5}{4} \cdot 50\%$

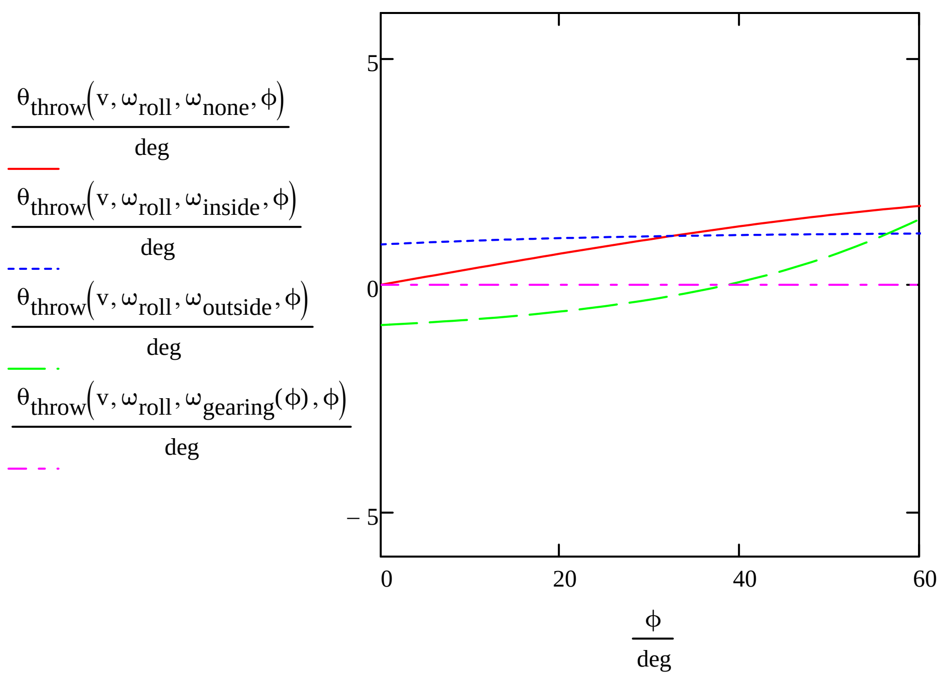
$\omega_{none} := 0$ $\qquad \omega_{inside} := -SRF\,\omega_{roll}$ $\qquad \omega_{outside} := SRF\,\omega_{roll}$ $\qquad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$



throw vs. cut angle for various types of 50% side-English, medium speed, follow/draw shots:

$v := v_{medium}$ $\qquad \omega_{roll} := \frac{v}{R}$ $\qquad \omega_{stun} := 0$ $\qquad SRF := \frac{5}{4} \cdot 50\%$

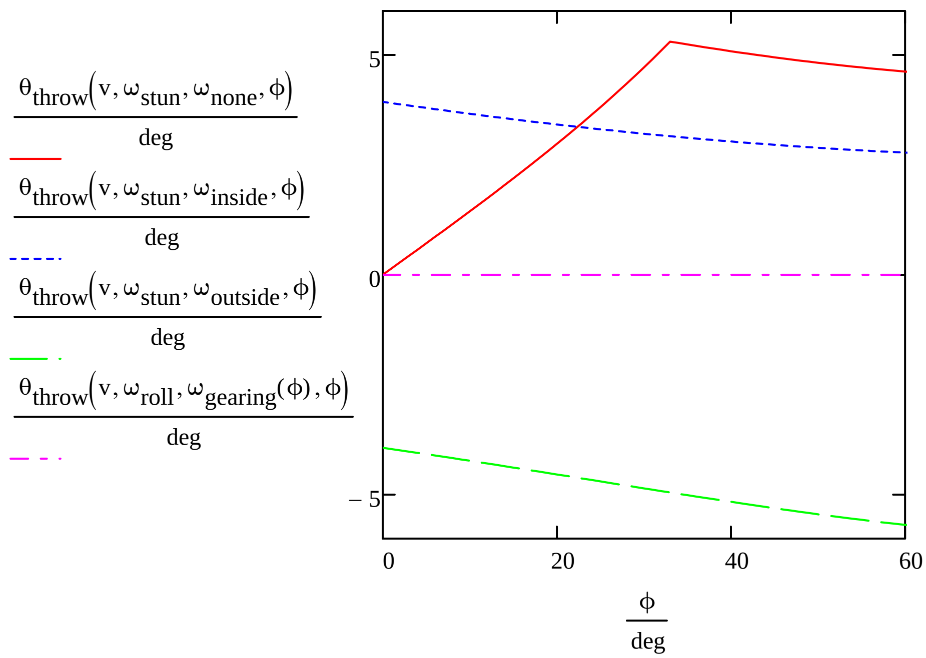
$\omega_{none} := 0$ $\qquad \omega_{inside} := -SRF\,\omega_{roll}$ $\qquad \omega_{outside} := SRF\,\omega_{roll}$ $\qquad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$



throw vs. cut angle for various types of 100% side-English, slow speed, stun shots:

$v := v_{slow}$ $\quad \omega_{roll} := \frac{v}{R}$ $\quad \omega_{stun} := 0$ $\quad SRF := \frac{5}{4} \cdot 100\%$

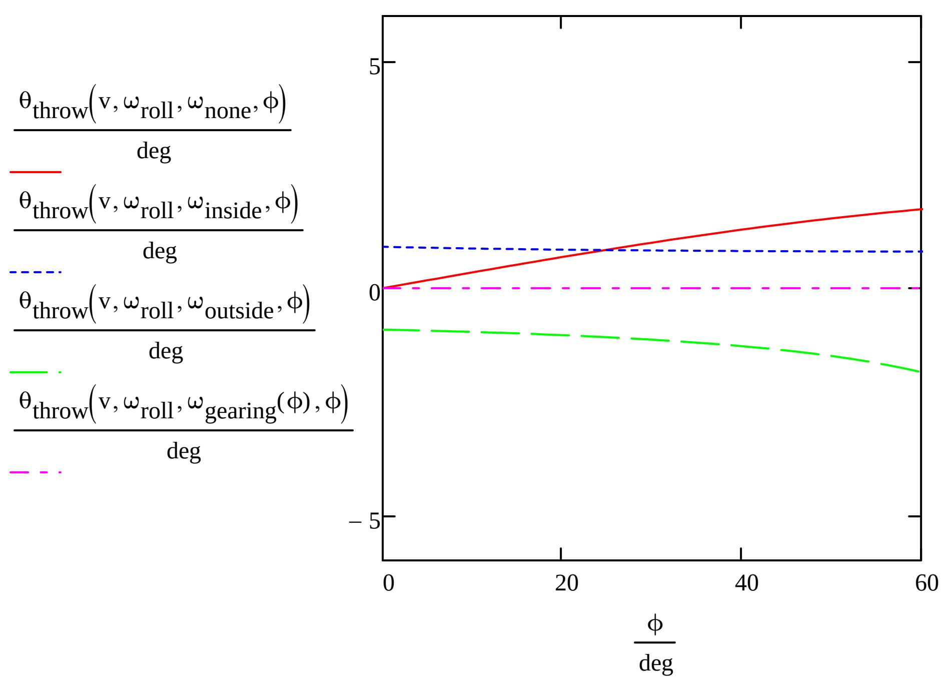
$\omega_{none} := 0$ $\quad \omega_{inside} := -SRF\,\omega_{roll}$ $\quad \omega_{outside} := SRF\,\omega_{roll}$ $\quad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$



throw vs. cut angle for various types of 100% side-English, medium speed, follow/draw shots:

$v := v_{medium}$ $\quad \omega_{roll} := \frac{v}{R}$ $\quad \omega_{stun} := 0$ $\quad SRF := \frac{5}{4} \cdot 100\%$

$\omega_{none} := 0$ $\quad \omega_{inside} := -SRF\,\omega_{roll}$ $\quad \omega_{outside} := SRF\,\omega_{roll}$ $\quad \omega_{gearing}(\phi) := \frac{v \cdot \sin(\phi)}{R}$



Effects of cling/skid/kick at different cut angles:

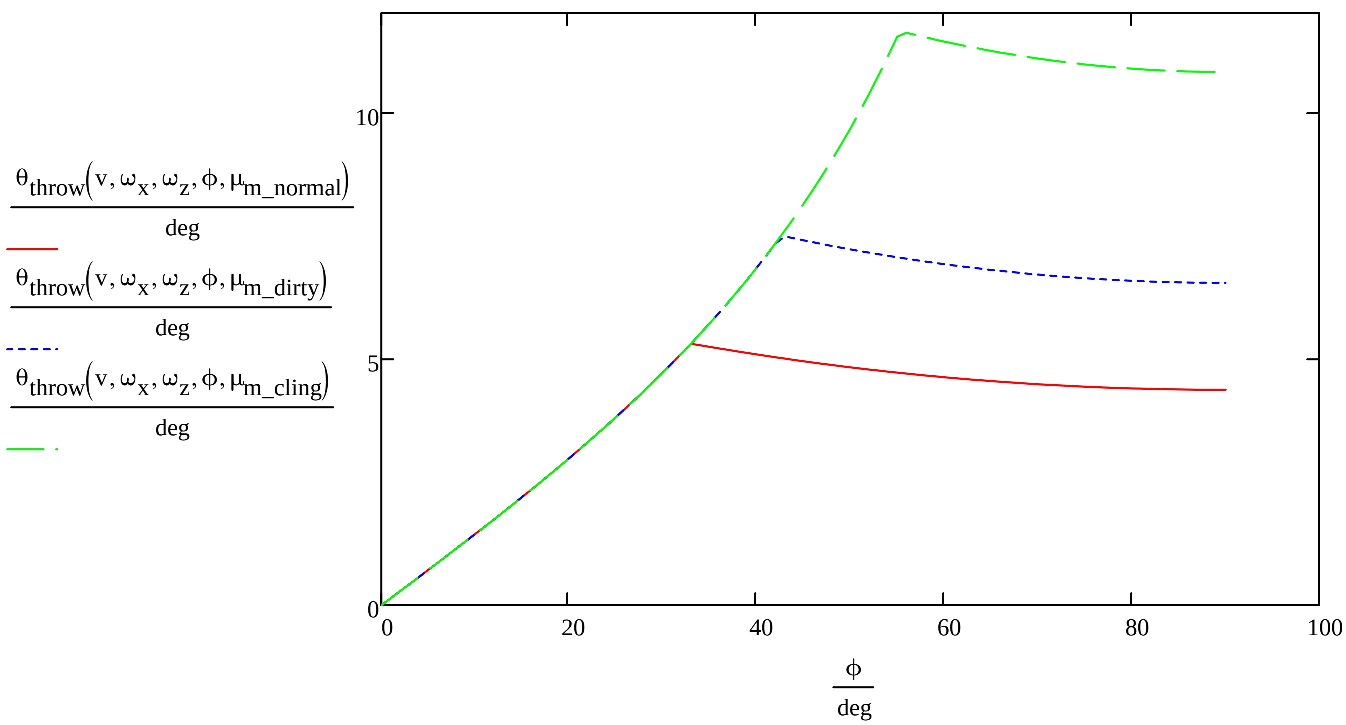
μ_m: friction multiplier (1: normal, >1: dirty conditions, up to 3: cling/skid/kick)

$$\theta_{throw}(v, \omega_x, \omega_z, \phi, \mu_m) := \operatorname{atan}\left[\frac{\mathrm{if}\left[\left(v_{rel}(v,\omega_x,\omega_z,\phi) = 0\right), 0, \min\left(\frac{\mu_m \cdot \mu\left(v_{rel}(v,\omega_x,\omega_z,\phi)\right)\cdot v\cdot\cos(\phi)}{v_{rel}(v,\omega_x,\omega_z,\phi)}, \frac{1}{7}\right)\cdot\left(v\cdot\sin(\phi) - R\cdot\omega_z\right)\right]}{v\cdot\cos(\phi)}\right]$$

Throw and cling/skid/kick is maximum for a slow, stun shot:

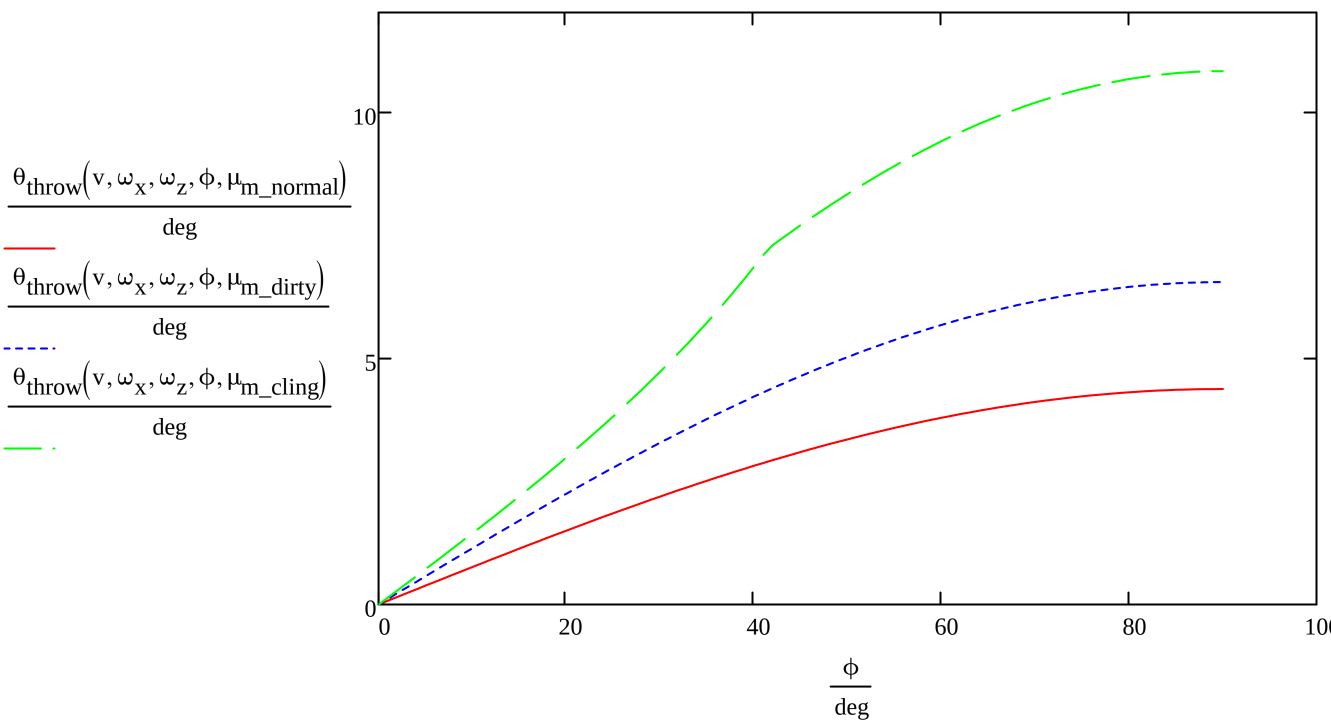
$v := v_{slow}$ $\qquad \omega_x := 0$ $\qquad \omega_z := 0$

$\mu_{m\_normal} := 1.0$ $\qquad \mu_{m\_dirty} := 1.5$ $\qquad \mu_{m\_cling} := 2.5$



For stun shots, the maximum amount of throw is normally at close to a 1/2-ball hit (30 degree cut angle); but with clingy conditions, maximum throw occurs at larger cut angles (50-60 degrees). For small cut angles (less than about 30 degrees), the amount of throw is the same regardless of the amount of friction.

For a slow rolling-CB shot, here's how cling/skid/kick affects shots: $\omega_x := \frac{v}{R}$



For rolling-CB shots, throw increases with cling at all cut angles, and is larger at larger cut angles.