TP A.6
The effects of ball inelasticity and friction on the 30° rule

supporting:
"The Illustrated Principles of Pool and Billiards"
http://billiards.colostate.edu
by David G. Alciatore, PhD, PE ("Dr. Dave")

originally posted: 2/7/05 last revision: 1/9/21

See TP A.4 and TP A.5 for background information and illustrations.

The cue ball is assumed to be rolling initially in the y direction (see the figure in TP A.4):

$$\vec{v} = v\hat{j} \qquad \vec{\omega} = -\frac{v}{R}\hat{i} \tag{1}$$

From Equation 7 in TP A.5, the normal component of the post-impact cue ball velocity is:

$$v_{n0} = \frac{(1-e)\cos(\phi)}{2}v \tag{2}$$

To find the tangential component, we need to know the direction of the friction impulse. This direction will be opposite from the direction of the relative sliding velocity of the point of contact (B) between the cue ball and the object ball, at impact. The total velocity of this point is:

$$\vec{v}_B = \vec{v} + \vec{\omega}\times\vec{r}_{B/O} = v\hat{j} + (-\frac{v}{R}\hat{i})\times R(-\sin(\phi)\hat{i} + \cos(\phi)\hat{j}) = v\hat{j} - v\cos(\phi)\hat{k} \tag{3}$$

Using Equation 4 in TP A.5, and eliminating the normal component of this velocity, we can express the relative sliding velocity vector for the point of contact:

$$\vec{v}_{rel} = v\left(\sin(\phi)\hat{t} - \cos(\phi)\hat{k}\right) \tag{4}$$

From Equation 8 in TP A.5, the normal impulse is given by:

$$\hat{F}_n = \frac{(1+e)\cos(\phi)}{2}mv \tag{5}$$

Therefore, the friction impulse can be expressed as:

$$\hat{F}_{fric} = -\mu_{balls}\hat{F}_n\frac{\vec{v}_{rel}}{\left|\vec{v}_{rel}\right|} = \frac{1}{2}\mu_{balls}mv(1+e)\cos(\phi)\left(-\sin(\phi)\hat{t} + \cos(\phi)\hat{k}\right) \tag{6}$$

Therefore, the tangential component of the friction impulse (reversing the sign to match the direction in TP A.5) is:

$$\hat{F}_{fric_t} = \frac{1}{2}\mu_{balls} mv(1+e)\sin(\phi)\cos(\phi) \tag{7}$$

and the vertical component is:

$$\hat{F}_{fric_z} = \frac{1}{2}\mu_{balls} mv(1+e)\cos^2(\phi) \tag{8}$$

Now we can use Equation 7 above and Equations 2 and 4 from TP A.5 to find the tangential component of the post-impact cue ball velocity:

$$v_{t0} = v\sin(\phi) - \frac{\hat{F}_{fric_t}}{m} = v\sin(\phi)\left(1 - \frac{1}{2}\mu_{balls}(1+e)\cos(\phi)\right) \tag{9}$$

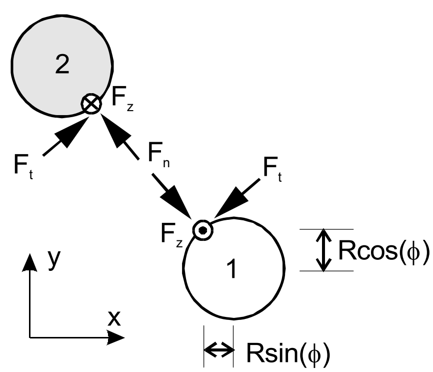
The vertical component of the friction impulse changes the angular momentum of the cue ball during impact. Refer to the figure below. The momentum about the x-axis changes according to:

$$I\omega_{xo} = I\omega_x + R\cos(\phi)\hat{F}_{fric_z} \tag{12}$$

and about the y-axis according to:

$$I\omega_{yo} = I\omega_y + R\sin(\phi)\hat{F}_{fric_z} \tag{13}$$

impact forces



Note: The tangential component of the friction impulse affects the z-axis spin of the ball, but this has negligible effect on the cue ball trajectory (see TP A.4).

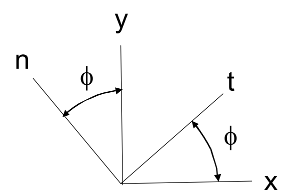
Therefore, using Equations 1 and 8, the post-impact angular speed about the x-axis is:

$$\omega_{xo} = \omega_x + \frac{R\cos(\phi)\hat{F}_{fric_z}}{\frac{2}{5}mR^2} = -\frac{v}{R} + \frac{5\mu_{balls}v(1+e)\cos^3(\phi)}{4R} = \frac{v}{R}\left(\frac{5}{4}\mu_{balls}(1+e)\cos^3(\phi) - 1\right) \quad (16)$$

and about the y-axis is:

$$\omega_{yo} = \omega_y + \frac{R\sin(\phi)\hat{F}_{fric_z}}{\frac{2}{5}mR^2} = 0 + \frac{5\mu_{balls}v(1+e)\sin(\phi)\cos^2(\phi)}{4R} = \frac{v}{R}\left(\frac{5}{4}\mu_{balls}(1+e)\sin(\phi)\cos^2(\phi)\right) \quad (17)$$

The normal and tangential speed components for the cue ball need to be projected into the x and y directions to be able to use the analysis in TP A.4. The figure below shows how the components are related.



Using Equations 2 and 9, the x and y components of the post-impact cue ball velocity can be expressed as:

$$v_{x0} = v_{t0}\cos(\phi) - v_{n0}\sin(\phi) = \frac{v}{2}\sin(\phi)\cos(\phi)\left(1 + e - \mu_{balls}(1+e)\cos(\phi)\right) \quad (18)$$

$$v_{y0} = v_{t0}\sin(\phi) + v_{n0}\cos(\phi) = \frac{v}{2}\left[\sin^2(\phi)\left(2 - \mu_{balls}(1+e)\cos(\phi)\right) + (1-e)\cos^2(\phi)\right] \quad (19)$$

Now that we know the cue ball post-impact velocity components (Equations 11, 12, and 13), we can use the results from TP A.4 to see the effect on the cue ball trajectory. Equations 24 and 25 from TP A.4 describe the cue ball trajectory:

$$x(t) = v_{xo}t - \frac{\mu g v_{Cxo}}{2v_{Co}}t^2 \quad (20)$$

$$y(t) = v_{yo}t - \frac{\mu g v_{Cyo}}{2v_{Co}}t^2 \quad (21)$$

From Equation 14 in TP A.4, the numerator terms are:

$$v_{Cxo} = v_{xo} - R\omega_{yo} \quad (22)$$

$$v_{Cyo} = v_{yo} + R\omega_{xo} \quad (23)$$

From Equation 15 in TP A.4, the denominator term is:

$$v_{Co} = \sqrt{v_{Cxo}{}^2 + v_{Cyo}{}^2} \tag{24}$$

From Equation 20 in TP A.4, the time required for the cue ball to begin rolling is:

$$\Delta t = \frac{2v_{Co}}{7\mu g} \tag{25}$$

From Equations 21 and 22 in TP A.4, the final cue ball velocity components are:

$$v_{xf} = \frac{1}{7}\left(5v_{xo} + 2R\omega_{yo}\right) \tag{26}$$

$$v_{yf} = \frac{1}{7}\left(5v_{yo} - 2R\omega_{xo}\right) \tag{27}$$

From Equation 23 in TP A.4, the final deflected cue ball angle is:

$$\theta_c = \tan^{-1}\left(\frac{5v_{xo} + 2R\omega_{yo}}{5v_{yo} - 2R\omega_{xo}}\right) \tag{28}$$

Here are typical values for the parameters used in the equations along with the MathCAD forms of the results:

$e := 0.94$	coefficient of restitution between balls
$\mu_{balls} := 0.06$	average coefficient of friction between the balls
$\phi := 30 \cdot deg$	half-ball hit
$\mu := 0.2$	coefficient of friction between the cue ball and table cloth:
$g := \frac{g}{\left(\frac{m}{s^2}\right)}$	acceleration due to gravity $g = 9.807$
$v := 2$	average pre-impact cue ball speed in m/s
$R := \frac{\frac{2.25}{2} \cdot in}{m}$	ball radius

initial cue ball velocity components:

$$\omega_{x0}(v,\phi,e,\mu_{balls}) := \frac{v}{R}\cdot\left[\frac{5}{4}\cdot\mu_{balls}\cdot(1+e)\cdot\cos(\phi)^3 - 1\right]$$ from Equation 16

$$\omega_{y0}(v,\phi,e,\mu_{balls}) := \frac{v}{R}\cdot\left[\left[\frac{5}{4}\cdot\mu_{balls}\cdot(1+e)\cdot\sin(\phi)\cdot\cos(\phi)^2\right]\right]$$ from Equation 17

$$v_{x0}(v,\phi,e,\mu_{balls}) := \frac{v}{2}\cdot\sin(\phi)\cdot\cos(\phi)\cdot\left[1+e-\mu_{balls}\cdot(1+e)\cdot\cos(\phi)\right]$$ from Equation 18

$$v_{y0}(v,\phi,e,\mu_{balls}) := \frac{v}{2}\cdot\left[\sin(\phi)^2\cdot\left[2-\mu_{balls}\cdot(1+e)\cdot\cos(\phi)\right]+(1-e)\cdot\cos(\phi)^2\right]$$ from Equation 19

v_C terms from TP A.4 used in several Equations:

$$v_{Cx0}(v,\phi,e,\mu_{balls}) := v_{x0}(v,\phi,e,\mu_{balls}) - R\cdot\omega_{y0}(v,\phi,e,\mu_{balls})$$ from Equation 22

$$v_{Cy0}(v,\phi,e,\mu_{balls}) := v_{y0}(v,\phi,e,\mu_{balls}) + R\cdot\omega_{x0}(v,\phi,e,\mu_{balls})$$ from Equation 23

$$v_{C0}(v,\phi,e,\mu_{balls}) := \sqrt{v_{Cx0}(v,\phi,e,\mu_{balls})^2 + v_{Cy0}(v,\phi,e,\mu_{balls})^2}$$ from Equation 24

time required for the cue ball to start rolling (cease sliding):

$$\Delta t(v,\phi,e,\mu_{balls}) := \frac{2\cdot v_{C0}(v,\phi,e,\mu_{balls})}{7\cdot\mu\cdot g}$$ from Equation 25

velocity components when the cue ball starts rolling in a straight line:

$$v_{xf}(v,\phi,e,\mu_{balls}) := \frac{1}{7}\cdot\left(5\,v_{x0}(v,\phi,e,\mu_{balls}) + 2\cdot R\cdot\omega_{y0}(v,\phi,e,\mu_{balls})\right)$$ from Equation 26

$$v_{yf}(v,\phi,e,\mu_{balls}) := \frac{1}{7}\cdot\left(5\cdot v_{y0}(v,\phi,e,\mu_{balls}) - 2\cdot R\cdot\omega_{x0}(v,\phi,e,\mu_{balls})\right)$$ from Equation 27

the final deflected cue ball angle:

$$\theta_c(v,\phi,e,\mu_{balls}) := \operatorname{atan}\left(\frac{5\cdot v_{x0}(v,\phi,e,\mu_{balls}) + 2\cdot R\cdot\omega_{y0}(v,\phi,e,\mu_{balls})}{5\cdot v_{y0}(v,\phi,e,\mu_{balls}) - 2\cdot R\cdot\omega_{x0}(v,\phi,e,\mu_{balls})}\right)$$ from Equation 28

x position of the cue ball during the curved trajectory:

$$x_c(t,v,\phi,e,\mu_{balls}) := v_{x0}(v,\phi,e,\mu_{balls})\cdot t - \frac{\mu\cdot g\cdot v_{Cx0}(v,\phi,e,\mu_{balls})}{2\cdot v_{C0}(v,\phi,e,\mu_{balls})}\cdot t^2$$ from Equation 20

x position of the cue ball during and after the curve trajectory:

$$x(t, v, \phi, e, \mu_{balls}) := \begin{array}{|l} \Delta T \leftarrow \Delta t(v, \phi, e, \mu_{balls}) \\ x_c(t, v, \phi, e, \mu_{balls}) \ \text{ if } \ t \le \Delta T \\ \left[x_c(\Delta T, v, \phi, e, \mu_{balls}) + v_{xf}(v, \phi, e, \mu_{balls})\cdot(t - \Delta T)\right] \ \text{ otherwise} \end{array}$$

y position of the cue ball during the curved trajectory:

$$y_c(t, v, \phi, e, \mu_{balls}) := v_{y0}(v, \phi, e, \mu_{balls})\cdot t - \frac{\mu\cdot g\cdot v_{Cy0}(v, \phi, e, \mu_{balls})}{2\cdot v_{C0}(v, \phi, e, \mu_{balls})}\cdot t^2$$

from Equation 21

y position of the cue ball during and after the curve trajectory:

$$y(t, v, \phi, e, \mu_{balls}) := \begin{array}{|l} \Delta T \leftarrow \Delta t(v, \phi, e, \mu_{balls}) \\ y_c(t, v, \phi, e, \mu_{balls}) \ \text{ if } \ t \le \Delta T \\ \left[y_c(\Delta T, v, \phi, e, \mu_{balls}) + v_{yf}(v, \phi, e, \mu_{balls})\cdot(t - \Delta T)\right] \ \text{ otherwise} \end{array}$$

Parameters used in the plot below:

$T := 4$ number of seconds to display

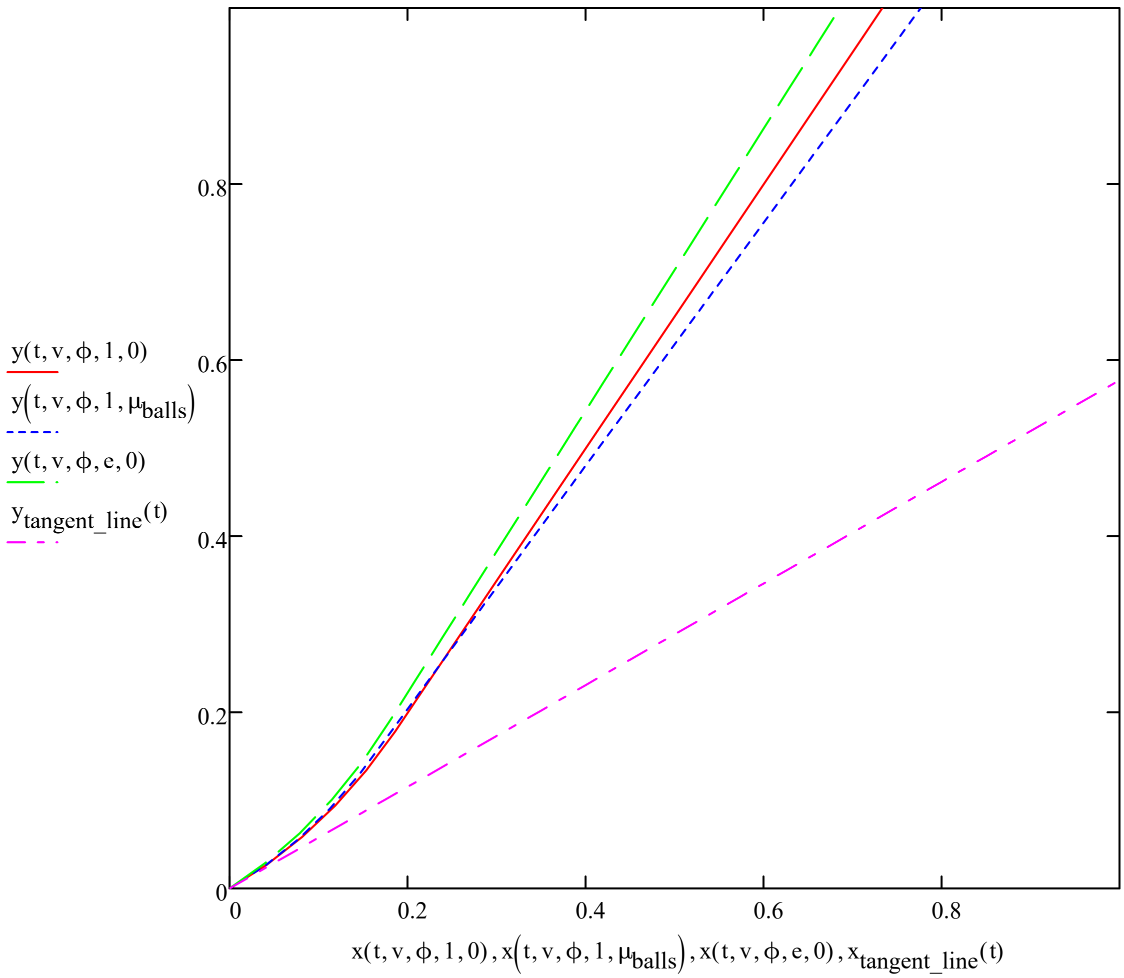
$t := 0, 0.05 .. T$ 0.1 second plotting increment

Equation for the tangent line:

$$x_{tangent\_line}(t) := \frac{t}{T}\cdot 2$$

$$y_{tangent\_line}(t) := x_{tangent\_line}(t)\cdot\tan(\phi)$$

half-ball hit with natural roll



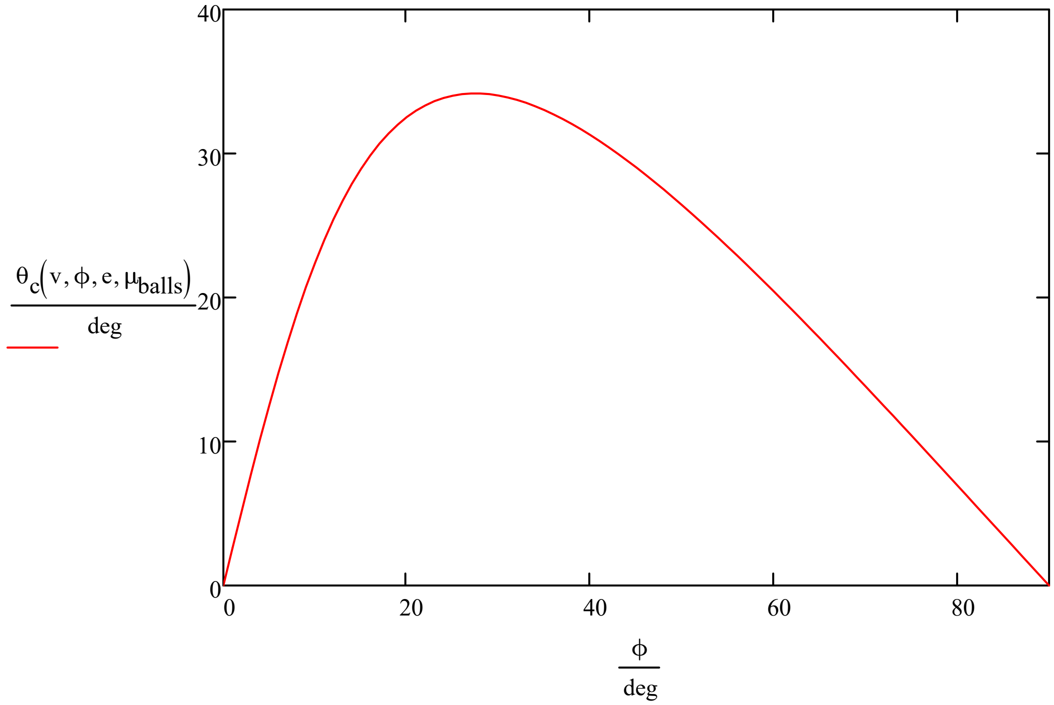
final deflected cue ball angle:

$\theta_c(v, \phi, 1, 0) = 33.67 \cdot deg$	with no ball inelasticity or friction
$\theta_c(v, \phi, 1, \mu_{balls}) = 35.907 \cdot deg$	with ball friction only
$\theta_c(v, \phi, e, 0) = 31.988 \cdot deg$	with ball inelasticity only
$\theta_c(v, \phi, e, \mu_{balls}) = 34.026 \cdot deg$	with ball inelasticity and friction

So friction lengthens the cue ball angle, and inelasticity shortens the cue ball angle. With both, the angle is changed only a little.

Here's how cue ball angle varies with cut angle (with ball ineleasticity and friction):

$$\phi := 0 \cdot deg, 1 \cdot deg .. 90 \cdot deg$$



Here is the maximum CB angle (θ) and corresponding cut angle (ϕ_{max}) for the various cases:

$\phi := 30 \cdot deg$ initial guess

with no ball inelasticity or friction:

$\theta(\phi) := \theta_c(v, \phi, 1, 0)$ $\phi_{max} := \text{Maximize}(\theta, \phi) = 28.126 \cdot deg$ $\theta(\phi_{max}) = 33.749 \cdot deg$

with ball friction only:

$\theta(\phi) := \theta_c(v, \phi, 1, \mu_{balls})$ $\phi_{max} := \text{Maximize}(\theta, \phi) = 26.443 \cdot deg$ $\theta(\phi_{max}) = 36.21 \cdot deg$

with ball inelasticity only:

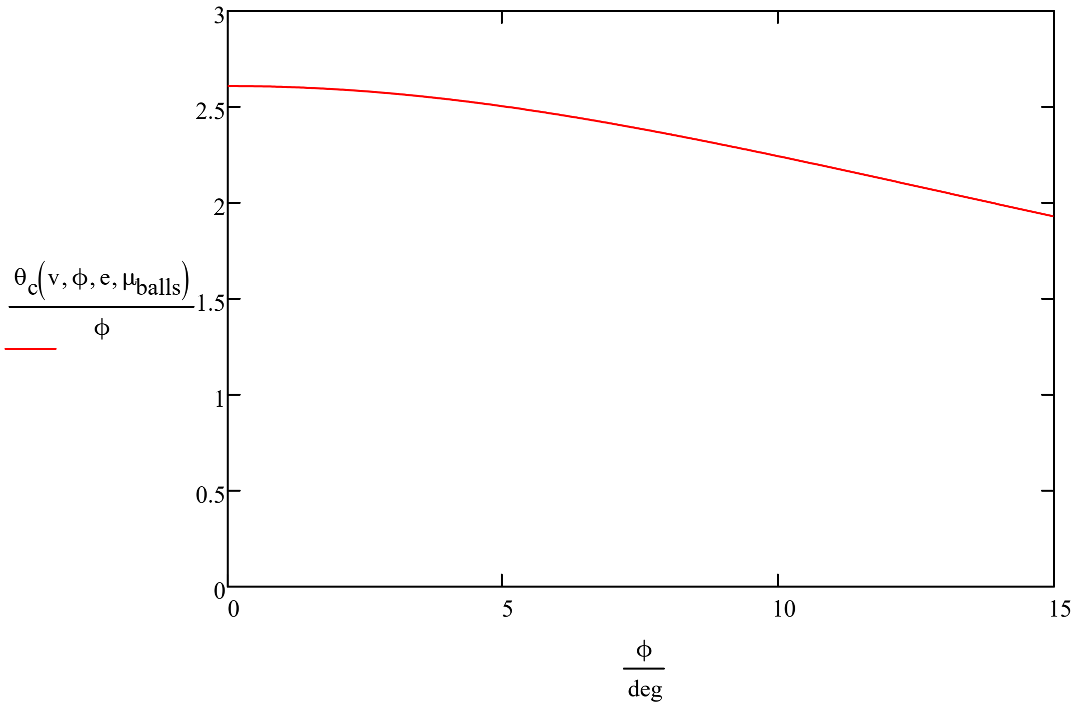
$\theta(\phi) := \theta_c(v, \phi, e, 0)$ $\phi_{max} := \text{Maximize}(\theta, \phi) = 28.995 \cdot deg$ $\theta(\phi_{max}) = 32.009 \cdot deg$

with ball inelasticity and friction:

$\theta(\phi) := \theta_c(v, \phi, e, \mu_{balls})$ $\phi_{max} := \text{Maximize}(\theta, \phi) = 27.468 \cdot deg$ $\theta(\phi_{max}) = 34.173 \cdot deg$

Here is how the ratio of CB angle to cut angle varies with cut angle (with ball inelasticity and friction):

$$\phi := 0.01 \cdot deg, 0.05 \cdot deg .. 15 \cdot deg$$



See TP B.13 to see how the flattness of this curve at low cut angles can be useful. See also:

https://billiards.colostate.edu/faq/cue-ball-control/where-cb-goes/